Flexible seasonal long memory and economic time series

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Abstract

We discuss specification, frequency domain estimation and application of a flexible seasonal long memory time series model based on fractional differencing. This type of model lends itself to seasonal unit root testing using standard distribution theory with null hypotheses of stationarity and nonstationarity. We apply Wald tests. We suggest periodogram regression estimation for simple models, which we evaluate positively using simulation. We apply the flexible model on widely used quarterly US GNP data. Approximate ML estimation shows the statistical significance of seasonal "overdifferencing" due to seasonal adjustment. Application to monthly shipping data for the Sound (1557-1783) shows the order of integration at frequency 0 and 1/12 about 0.5, with lower values at other frequencies. We use several graphical techniques to evaluate the estimation results in the frequency domain.

Keywords

long memory, fractional integration, seasonality, unit roots, frequency domain estimation, seasonal adjustment

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1 Introduction

This paper focuses on flexible modeling of long memory in the seasonal behavior of economic time series using frequency domain techniques. Following Gray et al. (1989) we consider a series to be long memory if its spectrum is infinite at a frequency, not necessarily the zero frequency. Long memory corresponds to slow (periodic) decay in the autocorrelations function. We apply a specific model in the class defined by Robinson (1994), which allows for fractional integration at several frequencies. This model encompasses earlier models by Ray (1991,1993) and Porter-Hudak (1990). The basic idea is to estimate separate fractional orders of integration at the frequencies relevant for seasonal behavior, j/s, $j = 1, \cdots, [s/2]$, where s is the number of observation periods per year, this in addition to the fractional integration parameter for the zero frequency, $j = 0$, and short memory (ARMA) parameters.

We also suggest an improvement of the periodogram regression procedure for the estimation of fractional integration parameters, proposed by Hassler (1994) for so-called ARFISMA models. In the empirical part of the paper we estimate the parameters of the long memory part and the short memory part of the model simultaneously using approximate frequency domain ML. The estimation technique is computationally only slightly more involved than nonlinear least squares.

Main results

We perform a Monte Carlo study on the periodogram estimators for the simplest case with fractional integration at frequency 1/4, where asymptotic distribution theory can easily be derived from the literature. Fitting the model spectrum to the sample spectrum in the neighborhood of a frequency of interest only adds extra inefficiency compared to using the full spectrum. Asymptotic theory based standard errors appear to give a good approximation of finite sample variation of the estimates of the integration parameter. Our modification of the periodogram estimation leads to smaller bias in relevant cases, without a higher variance. The asymptotic normality is achieved in moderately sized samples. Seasonal unit root tests based on this periodogram regression perform well. Standard theory does not apply to if one uses the estimation for processes with orders of integration smaller than -0.5. This corresponds to results obtained by Hurvich and Ray (1995) for the nonseasonal fractional model.

The empirical part of the paper shows that the model picks up "seasonal overdifferencing" due to seasonal adjustment in the growth rates of US GNP, earlier studied by Sowell (1992ab). An empirical analysis of shipping through the Sound (1557-1783) shows merits of the model for long series of monthly data. The standard (I(1)) model with seasonal unit roots in the sense of Hylleberg et al. (1990) is rejected *at all* frequencies. The hypothesis of weak dependence (I(0)) is rejected at several frequencies against fractional alternatives.

Structure of the paper

Section 2 contains a selective survey of related literature. In section 3 we formally introduce the model and some alternatives. We discuss the spectral density, the autocovariance function, the likelihood function and estimation. In section 4 we analyze two semi-parametric periodogram estimators in a Monte Carlo study. Section 5 contains empirical results for a series for US GNP, analyzed earlier by Sowell (1992a), which serves to illustrate the difference between estimation methods and the consequences of the model extension in a typical macroeconomic time series. In section 6 we apply the method to a historical century long monthly shipping series of considerable interest. This series shows the advantages of the model extension more clearly. Section 7 concludes.

2 Selective survey of related literature

After more than a decade of publishing work on economic time series modeling Mandelbrot (1972, p. 268) stated: “economic time series cannot be spectral synthesized, and their spectral analysis is a purely formal technique lacking concrete backing”. Indeed, later research has shown one has to interpret spectral decompositions, like the periodogram, with care if economic time series show signs of long memory, (Künsch(1986)) or extremely fat-tailed innovations (Klüppelberg and Mikosch (1993)). Mandelbrot considered these two aspects of economic time series as facts of life. He aptly labeled them Joseph effect (long memory) and Noah effect (too high probabilities of extreme outcomes) and considered them especially relevant for speculative price series. He did not deny that seasonal movements show up in spectral analysis, but:(op cit):“Seasonals do not count, because they are hardly hidden to begin with”.

Nevertheless, frequency domain techniques to analyze long memory in economic time series are thriving more than ever. The literature with corresponding asymptotic techniques for statistical inference is growing fast, see e.g. Robinson (1994) . It turns out that a lot of the traditional likelihood based asymptotic theory is applicable, especially concerning inference about the long memory parameter, as long as all computable sample autocovariances are used efficiently. Existence of second moments seems still an assumption we cannot do without in spectral analysis, which is, after all, a variance decomposition technique.

The most successful applications in economics of frequency domain based techniques for long memory in economics seem to have been in modeling inflation rates at comparatively high levels of aggregation in time and types of goods, see e.g. Hassler and Wolters (1995). Appropriate examples for real macroeconomic time series are harder to come by. This may have to do with the misspecification of the seasonal periodicity of the series. THe seasonal movements may show independent long memory aspects, overlooked in traditional long memory models. Many price series, even aggregate ones, may deserve better inspection of the seasonal component as well.

The literature on long memory applications in statistics and econometrics is huge. Beran (1992), Baillie (1994), and Robinson (1994b) provided useful surveys. Many types of estimators and inference have been suggested and applied. The literature on theory and applications of long memory models for seasonal behavior is still limited. We discuss a few recent developments below.

Seasonal extensions of long memory models

Analysis of long memory of seasonal movements was not favored by Mandelbrot. His favorite technique to analyze long memory, adjusted range analysis, or R/S analysis, later operationalized statistically by Lo (1991) is not at all suited for series with seasonal long memory, see Ooms (1994, section 7.4.7.3. for an empirical illustration).

Models for seasonal long memory have early been suggested by Abrahams and Dempster (1979), see Jonas (1983, p. 56). They were based on Mandelbrot’s favorite long memory model, continuous time fractional Gaussian noise, which we denote by fGn. This is a model with covariance function $\gamma_k = \frac{1}{2}\sigma^2 \left[|k+1|^{2h} - 2|k|^{2h} + |k-1|^{2h} \right]$, with h the *self similarity parameter* or *Hurst coefficient* at lag k and variance σ^2, $0 < h < 1$. This model has infinite spectrum at the zero frequency only, so it cannot capture very strong seasonal variation. Jonas (1983) generalized this model to allow for an infinite spectrum at the seasonal frequencies. He derived the spectral density and autocorrelation function, which we present below in section 3. Carlin et al. (1985) tried to combine nonseasonal and seasonal fractional Gaussian noise in an unobserved component model for economic time series. Due to technical difficulties in deriving the covariance function for this extended model, see Carlin and Dempster (1989, p. 10) and lack of information in the data set (16 years of monthly data of shipment values), they were not able to derive ML estimates, let alone try more flexible models.

A second model for long memory, which has become more popular in econometrics is the (discrete time) fractionally differenced model. It was suggested as an alternative to ARIMA models by Hosking (1981) and Granger and Joyeux (1980). A fractionally differenced time series x_t is defined as $(1-L)^d x_t = u_t$, with u_t Gaussian white noise and L the lag operator: $L^s x_t = x_{t-s}$ The difference operator, $(1-L)$, is raised to a fractional power, d, denoting the *fractional order of integration.* The long memory characteristics of x_t are in important aspects similar to those of fractional Gaussian noise with $h = d + \frac{1}{2}$. The autocorrelations ρ_k decay asymptotically like $|k|^{2d-1}$, whereas they decay like $|k|^{2h-2}$ for the fGn. This hyperbolic decay at high lags distinguishes series with long memory from series with short memory. It is the main characteristic for empirical identification.

This model lent itself to seasonal generalizations as well. Most practical research has been done on the fractional generalization of the seasonal ARIMA models suggested by Box and Jenkins (1970). Here the seasonal difference operator, $(1-L^s)$, is raised to a fractional power (s the number of observations per year). Porter-Hudak (1990) applied the seasonal generalization to US monetary aggregates using simple periodogram regression to estimate d. Ray(1991) generalized many statistical results available for the standard fractional differencing model to models involving $(1-L^s)$, and applied it to a short monthly revenue series.

Gray et al. (1989) suggested a so-called Gegenbauer ARIMA model involving the operator $(1-2uL+L^2)^{d_{\lambda_j}}$, with u an parameter to be estimated. Chung (1993) discussed estimation of u. Substitution of $\cos 2\pi\lambda_j$ for u leads to $(1-2\cos 2\pi\lambda_j L + L^2)^{d_{\lambda_j}} = ((1-e^{i2\pi\lambda_j}L)\cdot(1-e^{-i2\pi\lambda_j}L))^{d_{\lambda_j}}$, which shows this model is well suited for fractional modeling of so-called seasonal unit roots at frequencies λ_j. Take e.g., $u = 0$, $\lambda_j = \frac{1}{4}$, $e^{i2\pi\lambda_j} = i$, $(i^2 = -1)$ to see how it applies to the yearly frequency in quarterly data. Again, derivation of the covariance function encountered some technical difficulties (Gray et al. (1994)), but the spectrum is easily derived. Below, we use the parameterization by Gray et al. with fixed λ_js.

Beran (1993) generalized the exponential model of Bloomfield (1973), which is directly defined in the frequency domain, with long memory terms. The likelihood is defined for the periodogram of a series, multiplying the theoretical spectrum with independent exponential random variables with unit expectation. Beran therefore called it a fractional exponential model (FEXP). Estimation is done by generalized linear regression in the frequency domain. Usual asymptotic inference applies. Exponential time series models or estimation of the model spectrum using exponential distribution based techniques is rarely applied in economics. Shea (1991, p. 298) seems to be an exception.

Periodic fractionally integrated models, i.e. models where the fractional integration parameter varies from season to season, form an interesting alternative to fractional seasonal integration models. They are most easily interpreted in the context of the AR representation. Franses and Ooms (1995) applied the most simple form of this model to UK inflation rates with encouraging results. Distribution theory of the estimators and analysis of the corresponding time-dependent spectra and autocovariance functions still has to be developed, although some unpublished work in this area seems to exist, see Beran (1992), p. 407)

Seasonal unit root tests

Inference on the fractional order of integration in flexible seasonal long memory models can be applied to so-called seasonal unit root tests. Hylleberg et al. (1990) showed how to test for *integer* order of integration at different frequencies λ_j, j/s belonging to the "roots" of $(1 - z^s) = 0$ in a time series x_t, i.e. test $d_{\lambda_j} = 1$ against $d_{\lambda_j} = 0$ in $\prod_{j=1}^{s}(1 - e^{i2\pi j/s}L)^{d_{\lambda_j}} x_t = u_t$, using so-called *seasonal unit root* tests. They worked out the case $s = 4$, $(1 - L^4) = (1 + L)(1 - L)(1 + iL)(1 - iL)$, and a weakly dependent u_t in detail. Subsequent widespread application of this test has shown that the orders of integration differ across frequencies in many economic time series. This suggests that fractional generalizations along this line could be successful as well.

Limiting distributions of these test statistics are nonstandard and have to be worked out nearly on a case-by-case basis. Seasonal integration test statistics have so-called Dickey-(Hasza)-Fuller type limiting distributions, which depend on the presence of certain conditioning variables, like (seasonal) (breaking) trends, structural intervention dummies and the like, see e.g. Ghysels et al. (1994) or Ooms (1994, appendix 2.2). Canova and Hansen (1992) developed a test for $d_{\lambda_j} = 0$ against $d_{\lambda_j} = 1$, adapting a parameter stability test of Nyblom (1989). This test has similar problems.

Seasonal unit root test in the context of fractional models are easier to apply. Asymptotic critical regions of the LM tests developed by Robinson (1994a) do not depend on the presence of deterministic regressors. These test can be used to test both I(0) and I(1) types of null hypotheses against fractional alternatives. Limiting distributions of these test statistics are all χ^2. Robinson suggested that LR and Wald tests would have the same limiting distributions. Below we apply Wald tests, looking at t-values in unrestricted models, where trending regressors are not present.

Seasonal extension of periodogram regression

Periodogram (ordinary least squares) regression estimators of long memory parameters for simple fractional ARIMA models found widespread use in the analysis of economic time series following Geweke and Porter-Hudak (1983), see e.g. Hassler and Wolters (1995). Hassler generalized the procedure of Geweke and Porter-Hudak to allow estimation of parameters d_{λ_j} in what he called an ARFISMA model. Ooms (1994, section 7.4.7.3) , suggested a similar frequency by frequency estimation procedure for the orders of integration. He applied the procedure to a number of macroeconomic quarterly Dutch economic time series in order to interpret outcomes of traditional seasonal unit root tests and to check the theoretical effects of seasonal adjustment. We compare the methods of Hassler and Ooms in a simulation exercise. The regression tests are easy to compute and can be made reasonably robust.

We conclude this discussion of related literature by noting that the development of diagnostics is still underdeveloped. Only tests for residual autocorrelation abound, see e.g. Robinson (1994). Beran and Terrin (1994) examined a test for parameter stability. They showed the limiting distribution of a (generalized Chow) test statistic for the stability of h (or, for that matter, d,) against an alternative with an a priori defined number and timing of changes to be asymptotically χ^2 with degrees of freedom equal to the number of changes.

3 Flexible seasonal long memory

In this section we describe the model, and we discuss stationarity, invertibility, spectrum and computation of the covariance function. Next we discuss (approximate) ML estimation, inference and (seasonal) unit root testing.

Model, spectrum and covariances

We use the following model, adapted from Robinson (1994a), Ray(1991, 1993) and Hassler (1994). Hassler called it a "flexible ARFISMA" model. We call it a flexible (seasonal) ARIMA$(p, d, q)_s$ model, where the subscript s denotes the number of observations per year. It reads:

$$y_t = \beta' Z_t + x_t$$

$$D(L)x_t = u_t, \quad t = 1, 2, \ldots,$$

$$D(L) = (1 \quad L)^{d_0}(1 + L)^{d_{1/2}} \prod_{j=1}^{[(s \quad 1)/2]} \left(1 \quad 2\cos 2\pi j/sL + L^2\right)^{d_{j/s}}$$

$$\phi(L)u_t = \theta(L)\epsilon_t \tag{1}$$

where y_t is the observed real valued scalar variable, Z_t is an m vector of deterministic explanatory variables, like a constant, polynomial trends, seasonal dummies, trigonometric functions of time and cross products of these variables, x_t is the residual, β is an m vector of unknown parameters, d_{λ_j} is a real valued scalar parameter indicating the *fractional order of integration* of x_t *at frequency* $\lambda_j = j/s$, s is the number of observations per year, $\phi(z)$ is a "stationary" AR polynomial of order $p : \phi(z) = \phi_0 + \phi_1 z + \cdots + \phi_p z^p$, so that the roots of $\phi(z) = 0$ are outside the unit circle, $\theta(z)$ is an "invertible" MA polynomial of order $q : \theta(z) = \theta_0 + \theta_1 z + \cdots + \theta_q z^q$, so that the roots of $\theta(z) = 0$ are outside the unit circle, ϵ_t is a scalar Gaussian white noise "innovation" variable with variance σ_ϵ^2.

We normalize by taking $\phi_0 = \theta_0 = 1$. We take $d_{1/2} = 0$ for odd s. For $s = 3$ one obtains

$$D(L) = (1 \quad L)^{d_0}(1 + L + L^2)^{d_{1/3}}.$$

In most practical situations with economic time series s is even. For quarterly series, $s = 4$, one gets

$$D(L) = (1 \quad L)^{d_0}(1 + L)^{d_{1/2}}(1 + L^2)^{d_{1/4}}.$$

The operator $D(L)$ is defined for $\ \ 0.5 \leq d_{j/s} < 0.5$ by convolution of the expansions for its separate components, which read, see Hassler (1994):

$$(1 \quad L)^d = 1 \quad dL \quad \frac{d(1 \quad d)}{2}L^2 \quad ... = d(L) = \sum_{k=0}^{\infty} d_k L^k$$

$$d_k = \frac{(k \quad d)}{(k+1) \ (\ d)} \quad d_k \sim \frac{k^{\ d \ 1}}{(\ d)} \text{ as } k \to \infty \tag{2}$$

$$(1 + L)^{d_{1/2}} = 1 + d_{1/2}L \quad \frac{d_{1/2}(1 \quad d_{1/2})}{2}L^2 + ... = D_{1/2}(L) = \sum_{k=0}^{\infty} D_{1/2,k}L^k$$

$$D_{1/2,k} = (\ 1)^k d_k \tag{3}$$

$$(1 \quad 2\cos 2\pi(j/s)L + L^2)^{d_{j/s}} = D_{j/s}(L) = \sum_{j=0}^{\infty} D_{j/s,k} L^k$$

$$D_{j/s,k} = \sum_{l=0}^{k} d_{k \quad l} d_l \cos\left[2\pi j/s(k \quad 2l)\right] \tag{4}$$

where $(.)$ denotes the gamma function, which is defined for negative arguments by the recursion $z \quad (z) = \quad (z+1)$, or directly by $(z) = \pi/\left[\sin(\pi z) \cdot (\quad z \quad (\quad z))\right]$. Another notation is $(z) = (z \quad 1)!$. Computation of the expansions is done recursively. Extension to values of $d_{j/s}$ outside $[\quad 0.5, 0.5)$ is straightforwardly done by ordinary differencing or integration.

Note that the generating function $C(z)$ of the MA(∞) expansion of $D(L)$ equals $D^{\quad 1}(z)$, which is obtained by reversing the sign of the integration parameters in (2)-(4). The economically interesting impulse responses can thus be computed directly.

$D(L) = (1 + L^2)^{d_{1/4}}$ is an instructive case. Evaluation of the expansion can follow (3) substituting L^2 for L or one can use (4) directly, substituting $j = 1$, $s = 4$, which is easily seen to result in zero coefficients for odd k. We consider this case in more detail below.

The process x_t is *stationary* if $d_{j/s} < 0.5, \forall j$. It is *invertible* if $d_{j/s} > \quad 1$, $j = 0$. Odaki (1993) proved invertibility for this region. Presumably the same region applies at least also to $j/s = 1/2, 1/4$, noting the expressions above. Gray et al. (1989) proved invertibility for the region $d_{j/s} > \quad 0.5, \forall j$.

Other cases of interest are:

$D(L) = (1 \quad L)^{d_0}$, the model reduces to a nonseasonal fractionally integrated ARMA model for which acronyms FARMA and ARFIMA have been used. Sowell(1992ab) applied this model to quarterly log US GNP and estimated d_0 around 0.5.

$D(L) = (1 \quad 2\cos 2\pi j/sL + L^2)^{d_{j/s}}$, a reduction to a "restricted Gegenbauer" ARMA process. Gray et al. (1989) applied the model to the yearly sunspot data (1749-1924), estimated j/s (a free parameter in their set up) around 1/11 and $d_{j/s}$ around 0.3.

$D(L) = (1 \quad L^{12})^d$, "seasonal fractional integration", put $s = 12$, $d_0 = d_{1/s} = \cdots = d_{1/2} = d$. Porter-Hudak (1990) used this model on monthly US monetary aggregates, and estimated values for d around 0.5.

$D(L) = (1 \quad L^3)^{D_3}(1 \quad L^{12})^{D_{12}}$, put $s = 12$, $d_0 = d_{1/3} = D_3 + D_{12}$, and $d_{1/12} = d_{1/6} = d_{1/4} = d_{5/12} = d_{1/2} = D_{12}$. Ray (1991, 1993) used this model with an additional $\phi(L)$ of order 1, which she labeled SFARMA $(1,0,0) \times (0, D_3, 0)_3 \times (0, D_{12}, 0)_{12}$ (in analogy with seasonal Box-Jenkins models) to a monthly revenue series and estimated D_3 around 0.3 and D_{12} around 0.6. Reparameterization from our model to these SFARMA models is also possible: Note e.g. that $D(L) = (1 + L^2)^{d_{1/4}}$ can be written as $1 \quad L^2)^{D_2}(1 \quad L^4)^{D_4}$ by putting $D_4 = \quad D_2 = d_{1/4}$. It depends on the underlying model which parameterization is more convenient for estimation.

Memory characteristics

The *spectrum* of x_t equals, cf. Gray et al. (1989) for $\quad \pi < \omega \leq \pi$:

$$f_x(\omega) = \left|D(e^{i\omega})\right|^{\quad 2} f_u(\omega) = \frac{\sigma_\epsilon^2}{2\pi}\left|D(e^{i\omega})\right|^{\quad 2}\left|\phi(e^{i\omega})\right|^{\quad 2}\left|\theta(e^{i\omega})\right|^2,$$

$$\left|D(e^{i\omega})\right|^2 = |2(1 \quad \cos\omega)|^{d_0}\, |2(1+\cos\omega)|^{d_{1/2}} \prod_{j=3}^{[(s \quad 1)/2]} |2\,(\cos 2\pi j/s \quad \cos\omega)|^{2d_{j/s}}. \quad (5)$$

$f_x(\omega)$ is clearly infinite for $\omega = 2\pi j/s$ if $d_{j/s} > 0$. x_t is thus "long memory" if $\exists\, j : d_{j/s} > 0$.

All processes with positive $d_{j/s}$ posses a so-called *hyperbolic spectrum*, (terminology by Mandelbrot (1969)), but at different frequencies. We consider three representative cases.

For the simple nonseasonal process $x_t = (1 \quad L)^{\ d_0} u_t$, with $D(z) = (1 \quad z)^{d_0}$ one has $\left|D(e^{i\omega})\right|^2 = |2\,(\sin(\omega/2))|^{2d_0}$, so that $f_x(\omega, d_0) \sim f_u(\omega)\omega^{\ 2d_0}$ as $\omega \to 0$.

For the process $x_t^{'} = (1 + L^2)^{\ d_{1/4}} u_t$, with $D(z) = (1+z^2)^{d_{1/4}}$ we get $\left|D(e^{i2\pi\omega})\right|^2 = |2\,(\cos\omega)|^{2d_{1/4}}$, so that $f_{x'}(\omega + \pi/2, d_{1/4}) \sim f_u(\omega)(2(\omega+\pi/2))^{\ 2d_{1/4}}$ as $\omega \to \quad \pi/2$, and $f_{x'}(\omega \quad \pi/2, d_{1/4}) \sim f_u(\omega)(2(\omega \quad \pi/2))^{\ 2d_{1/4}}$ as $\omega \to \pi/2$.

For the process $x_t^{'} = (1 \quad L^2)^{\ d} u_t$ with "seasonal integration" simultaneously at frequencies 0 and $\frac{1}{2}$, i.e. the case with $D(z) = (1 \quad z^2)^d = (1+z)^d(1 \quad z)^d$, we have $\left|D(e^{i2\pi\omega})\right|^2 = |2\,(\sin\omega)|^{2d} = |(1 \quad \cos 2\omega)|^d$, so that $f_{x''}(\omega, d) \sim f_u(\omega)(2\omega)^{\ 2d}$ as $\omega \to 0$ and $f_{x''}(\omega, d) \sim f_u(\omega)(2\,(\pi \quad \omega))^{\ 2d}$ as $\omega \to \pi$. Note that $f_{x'}(\omega, d_{1/4}) = f_{x''}(\omega \quad \pi/2, d)$ if $d_{1/4} = d$, and that $f_{x'}\left(\omega, d_{1/4}\right) \sim 4 f_x(\omega, d_0)$, $\omega \to 0$ if $d_{1/4} = d_0$. The shapes of these spectra near the singular points are the same for equal orders of integration, but the levels differ.

The spectrum becomes zero at $\omega = 2\pi j/s$ if $d_{j/s} < 0$. The series is then called "over differenced", "anti-persistent" or "intermediate memory". The log spectrum, which plays a role in several estimation methods, does not exist at $\omega = 2\pi j/s$ if $d_{j/s} \neq 0$.

The *autocorrelation function* $\rho_k = \gamma_k/\gamma_0$ of x_t is only known in special cases. For $D(z) = (1 \quad (\ 1)^r z^s)^d$, $r = 0, 1$, $s = 1, 2, \ldots$, $|d| < 0.5$ and u_t white noise one has, see e.g. Adenstedt (1974, Lemma 5.1) Ray (1991, Theorem 3.1), Gray et al. (1989, Theorem 3):

$$\rho_k = \frac{(\ 1)^{r(k/s)} \ (\ d+1)\ (k/s+d)}{(k/s \quad d+1)\ (d)}, \quad k = is,\ i = 0, \pm 1, \pm 2, \ldots$$

$$\rho_k = 0 \ \text{ otherwise}$$

$$\rho_k \sim \frac{(\ d+1)}{(d)} (\ 1)^{r(k/s)} (k/s)^{2d \ \ 1} \ \text{ as } k \to \infty,\ k/s \text{ integer}. \quad (6)$$

The *variance* is given by $\gamma_0 = \sigma_u^2 \cdot \ (\ 2d+1)/\{\ (\ d+1)\}^2$.

Gray et al. (1994) obtained the following "high lag" autocorrelation function for the process with $D(z) = (1 \quad 2\cos 2\pi j/sL + L^2)^d$:

$$\rho_k \simeq \cos(kj/s)\,(k/s)^{2d \ \ 1} \text{as } k \to \infty$$

where $\simeq$ denotes equality in the limit up to an unknown finite nonzero constant, or, more generally, a so-called slowly varying function. Note the equality with (6) for $j/s = 1/4$, where the proportionality constant is known.

Sowell (1992b) derived an explicit expression for the covariance function for the FARMA model described above with finite order $\theta(z)$ and finite order $\phi(z)$, with p distinct roots of $\phi(z) = 0$.

For (continuous time) *seasonal fractional Gaussian noise* one has *spectrum*

$$f_{x,h}(\omega) \propto |(1 - \cos s\omega)| \cdot \sum_{n=-\infty}^{\infty} |n + s\omega/(2\pi)|^{-2h-1}, \tag{7}$$

and *correlation function*

$$\rho_{k,h} = \frac{1}{2}\left[|ks+1|^{2h} - 2\,|ks|^{2h} + |ks-1|^{2h}\right], \tag{8}$$

see Carlin and Dempster (1989, p. 9). Note that $\sum_{n=-\infty}^{\infty} |n + \omega/(2\pi)|^{-2} = 2\pi^2/\,|(1-\cos\omega)|$, so that a constant white noise spectrum results for $h = 0.5$, $d = 0$, in correspondence with $\rho_k = 0$, $k > 0$. One gets the long memory property $\rho_{k,h} \simeq (k/s)^{2h-2}$ as $|k| \to \infty$, and the hyperbolic spectrum $f_{x,h}(\omega) \simeq (s\omega)^{1-2h}$ as $|\omega| \to 0$, i.e. the model was designed for, see Beran (1992, p.406). In the simple case, $s = 1$, it is known (Carlin et al. (1985)) that the spectrum (7) has lower power at the high frequencies than the spectrum of the corresponding process $x_t = (1-L)^{-d}u_t$, with u_t white noise and $d = h - 0.5$.

Numerical illustration and approximation

In many cases (e.g. processes with integration at frequencies outside $0, 1/2$, and $1/4$) one still does not have closed form solutions for the autocovariance function. However, numerical approximations are possible. These approximations are useful in the estimation stage if exact computation of the autocovariance function (necessary in ML estimation) is impossible or computationally unattractive.

We can approximate the covariance function by numerical integration over the spectrum, see e.g. Granger and Newbold (1986, p. 46)

$$\gamma_k = \int_{-\pi}^{\pi} f_x(\omega)\, e^{i\omega k} \mathrm{d}\omega, \tag{9}$$

which is not trivial because of the singularities at $\omega = 2\pi j/s$, which have to be taken into account. Another way to compute the covariances is by inverting the set of Yule-Walker equations where one truncates the AR(∞) expansion of the model at a sufficiently high lag.

We illustrate the previous formulae for spectrum and autocovariance function and some of their numerical approximations for the long memory processes $x_t^{'} = (1+L^2)^{-1/3}u_t$ and $x_t'' = (1-L^2)^{-1/3}\, u_t$, with u_t white noise ($p = q = 0$).

Insert Figure 1

Figure 1 contains the logarithms of the theoretical spectra from (5). The logarithmic transformation helps to show the differences more clearly. The processes have been scaled to have the same variance. The hyperbolic character of the spectra derived above shows clearly. The spectrum of $x_t^{'}$, $f_{x'}(\omega, 1/3)$, has a pole at frequency 1/4. The spectrum of $x_t^{''}$, $f_{x''}(\omega, 1/3)$, has poles at 0 and 1/2. The spectrum of the corresponding fGn process (7) with $s = 2$ and $h = 5/6$ behaves similarly to $f_x(\omega, 1/3)$ near the poles but is somewhat smaller away from these. The approximation of $f_{x'}(\omega, 1/3)$ by Hassler is discussed in more detail in section 4 below. One sees it is lower away from the poles.

The log spectra for these simple fractional ARMA models for other values of $d \in (-0.5, 0.5)$ are proportional to the ones presented for $d = 1/3$, see (5). This is not the case for the fGn process for other values of $h \in (0,1)$. There the proportionality is only observed near the zeros and poles.

Insert_Figure_2

Figure 2 presents the correlation functions for the process $x_t^{'} = (1+L^2)^{\ \ 1/3} u_t$ and the short memory AR(1) process $x_t^{''}$ with $\ 1+0.5L^2)\, x_t^{''} = u_t$, $(\phi(z) = (1{+}0.5z^2),\ D\,(z) = 1)$ in one plot. The processes $x_t^{'}$ and $x_t^{''}$ have correlations ρ_k equal up to lag 3, $\rho_1 = 0$, $\rho_2 = \quad 0.5$, $\rho_3 = 0$. The next, ρ_4, equals 0.4 and 0.25 respectively. The extremely slow decay, with a cycle of period 4, comes out clearly for the long memory process $x_t^{'}$.

The other two series in the plot provide an idea about the accuracy of simple numerical approximations of a long memory covariance function. The first approximation uses the AR(∞) expansion (4) truncated at lag 100. We compute the autocovariance from the (inverted) Yule-Walker equations. For ρ_2, ρ_4 one gets -0.46 and 0.35 respectively, to be compared with the theoretical values of -0.5 and 0.4. For higher lags the approximation gets progressively worse. However, this procedure works better than simple numerical integration of the spectrum using a piecewise constant function evaluated at 1000 equidistant points on the interval $(\quad \pi, \quad \pi/2) \cup (\quad \pi/2, \pi/2) \cup (\pi/2, \pi]$ where one gets values of $\quad$ 0.43 and 0.31. The presence of poles in the spectrum make the latter technique hazardous for positive values of d.

Given that we know the correlation function of the long memory process in this case, we can also compute a finite order AR with an exactly matching correlation function up to lag 100. The differences between the resulting AR(100) approximation and the truncated AR(∞) expansion (4) are small at low lags. From (4) we get exact values for the AR coefficients $D_{1/4,2}$, $D_{1/4,4}$, $D_{1/4,6}$ of 0.333, -0.111 and 0.0617, whereas the AR(100) approximation gives 0.336, -0.113 and 0.0630.

One can make similar comparisons for other values of d. The AR approximation works better for positive d. For negative d, the case of "overdifferencing", the MA approximation is more appropriate, especially if $d < \quad 0.5$. The integration using the spectrum works markedly better for negative d, since the spectrum does not have poles in that case. The covariance function approaches zero much faster for negative d, see (2) .

This example shows one has to take care in choosing the right approximation method, which may differ from case to case. Naturally, one has to use as many lags as possible to get reasonable approximations of long memory models. This is especially relevant in estimation. We discuss estimation next.

Estimation and inference for the memory parameters

The exact or "unconditional " Gaussian log-likelihood function for the T vector of observations x, from the stationary process $\{x_t\}$, $t = 1, 2, \ldots, T$ reads:

$$L_T(\psi) = \quad \frac{T}{2} \ln 2\pi \quad \frac{1}{2} \ln \det\left(Q(\psi)\right) \quad \frac{1}{2} x' Q(\psi)^{\ \ 1} x \tag{10}$$

where $Q(\psi)$ is a Toeplitz matrix containing the autocovariances: $\{Q_{vw}\} = \gamma_{|v \quad w|}$, which is a function of the unknown parameter vector $\psi = \{\sigma_\epsilon^2, d_0, \ldots, d_{1/2}, \phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)$. We assume $\quad 0.5 \le d_{j/s} < 0.5$. T is an integer multiple of the number of observations per year, s.

Sowell (1992ab) maximized (10) directly. This is completely analogous to exact ML estimation of ARMA models, which has been implemented in popular computer packages like SAS, see e.g. Granger and Newbold (1986, section 3.5). Since one needs exact expressions for the autocovariance function, this direct technique is only applicable for s up to 2. Computation is not too time-consuming if one uses the Toeplitz structure in the Choleski decomposition of $Q\ ^{1}$, which simplifies evaluation of the likelihood. The Levinson-Durbin algorithm is numerically stable and reasonably fast. Storage requirements are considerable for large T, though.

For the general model one can apply different approximations to the likelihood function, which are all aimed at a diagonalization of $Q\ ^{1}$. We employ a frequency domain approximation using (9) [1]

$$\ln\det Q(\psi) \sim \sum_i \ln 2\pi f_x(\omega_i,\psi) \quad x'Q(\psi)\ ^{1}x \sim \sum_i \frac{I_T(\omega_i)}{f_x(\omega_i,\psi)} \tag{11}$$

where the summation runs over the Fourier frequencies $\omega_i = 2\pi i/T$, $i \in \{1,2,...,T|i \neq jT/s\}$, with the *periodogram* $I_T(\omega)$ evaluated as:

$$I_T(\omega) = \frac{1}{2\pi T}\left|\sum_{t=1}^{T} x_t e^{it\omega}\right|^2. \tag{12}$$

Harvey (1989, sections 4.3.1 and 5.5) discussed frequency domain estimation using approximation (11) in detail. Robinson (1994b, section 3.2) discussed (11) and several other approximations in the context of fractional models. We treat the behavior of the periodogram at the singular points $\omega_i = 2\pi i/T$, $i = jT/s$ below in a discussion of estimation of the seasonal means. The periodogram values can be interpreted in different ways. It is clear that $I_T(0) = (2\pi)\ ^{1}T\,|\overline{x}_T|^2$ is entirely determined by the mean. The periodogram values at other frequencies are in general functions of all available autocorrelations: $I_T(\omega) = (2\pi)\ ^{1}\sum_{k=\ T+1}^{T\ 1}\widehat{\gamma}_k\cos(k\omega)$, with $\widehat{\gamma}_k = T\ ^{1}\sum_{t=1}^{T\ |k|}(x_t\ \ \overline{x}_T)(x_{t\ \ k}\ \ \overline{x}_T)$, see e.g. Granger and Newbold (1986). Another equivalent expression is given by

$$I_T(\omega) = \frac{1}{2\pi T}\left(\left(\sum_{t=1}^{T} x_t\cos(t\omega)\right)^2 + \left(\sum_{t=1}^{T} x_t\sin(t\omega)\right)^2\right) \tag{13}$$

which shows a "trigonometric regression coefficient" interpretation of the periodogram, see e.g. Anderson (1971, p. 380). Note that $(2/T)\sum_{t=1}^{T} x_t\cos(t\omega)$ is the coefficient obtained by regressing x_t on $\cos(t\omega)$, $0 < \omega < \pi$.

Approximating $\sum_i \ln 2\pi f_x(\omega_i,\psi)$ in (11) by $T\ln\sigma_\epsilon^2$ and concentrating out σ_ϵ^2 leads to an equivalent minimization problem for the white noise variance:

$$\min_{\widetilde{\psi}}\widehat{\sigma}_\epsilon^2 = \frac{2\pi}{T}\sum_i \frac{I_T(\omega_i)}{g_x\left(\omega_i,\widetilde{\psi}\right)} \tag{14}$$

where $g_x(\omega_i,\widetilde{\psi}) = \ 2\pi/\sigma_\epsilon^2)\, f_x(\omega_i,\psi)$, and $\left\{\widetilde{\psi},\sigma_\epsilon^2\right\} = \psi$. The parameter σ_ϵ^2 has the interpretation of a one-step-ahead *prediction error variance,* see Harvey (1989, section 5.5). Fox and Taqqu (1986, p. 522 Remark 1) studied estimator (14) in the context of zero frequency long memory processes. They proved it to be $\sqrt{T}$ consistent, asymptotically efficient and

[1]The elements of $Q\ ^{1}$ can be evaluated directly as the inverse autocovariances, i.e. the autocovariances of the process with spectrum $1/f_x(\omega)$, see Fox and Taqqu (1986, p. 518).

normally distributed. We apply it below and compare it with results from exact maximum likelihood.

One can also minimize using both terms in (11), which leads to

$$\min_{\widetilde{\psi}} \left[\sum_i \ln g_x \left(\omega_i, \widetilde{\psi} \right) + T \ln \left(\frac{2\pi}{T} \sum_i \frac{I_T(\omega_i)}{g_x \left(\omega_i, \widetilde{\psi} \right)} \right) \right] \tag{15}$$

Boes et al. (1989) provided Monte Carlo evidence which favored using (15) over (14) in estimating a fractional ARIMA$(1, d, 1)$ model in small samples. Both estimators are nonlinear of course. Sowell (1992b) discussed the choice of starting values for the parameters in the estimation process. Robinson (1993) suggested to use the estimator (15) in the neighborhood of the frequencies of interest only, and derived results under weaker assumptions than Gaussianity. The asymptotic covariance matrix of the parameter estimates is computed using numerical approximations of the inverse Hessian of the respective approximations of the likelihood function.

Asymptotic behavior of the (seasonal) sample mean

So far we have not discussed estimation of β in (1), i.e. the parameter vector of the constant and seasonal dummies. The estimators for these parameters do not behave as usual and converge at a slower than usual rate if $d_{j/s} > 0$. Here we discuss the asymptotic behavior of the sample mean and the "trigonometric coefficients" from the ordinary least squares regression

$$y_t = \beta_0 + \beta_1(\ \ 1)^t + \beta_2 \cos(\pi t/2) + \beta_3 \sin(\pi t/2) + x_t \quad (t = 1, \ldots, T) \tag{16}$$

of the observations y_t on a constant and the orthogonal *trigonometric regressors* corresponding to frequencies $\frac{1}{2}$ and $\frac{1}{4}$. x_t is now the error term of the regression. Note that the regressors are collinear with the 4 *seasonal dummy variables* for quarterly data, so that properties for the *seasonal sample means* are easily derived from the results for this regression. The periodogram ordinates that we leave out in the approximate ML estimation method are simple functions of these regression coefficients: Note from (13) that $I_T(0) = (T/(2\pi))\, \widehat{\beta}_0^2$, $I_T(\pi) = (T/(2\pi))\, \widehat{\beta}_1^2$, and $I_T(\pi/2) = (T/(8\pi)) \left(\widehat{\beta}_2^2 + \widehat{\beta}_3^2 \right)$.

The asymptotic behavior of the sample mean, $\widehat{\beta}_0 = \overline{y}_T$, is well known. From the correlation function (8) for simple fGn one obtains $\mathrm{var}(\overline{y}_T) \sim \sigma_y^2 T^{2h\ \ 2}$, $T \to \infty$. For a short memory process one has $\mathrm{var}(\overline{y}_T) \sim 2\pi f_y(0) T^{\ \ 1}$, $T \to \infty$, with $f_y(0)$ the spectrum of y_t at the zero frequency. This behavior of the *variance-time function*, i.e. the variance of the (sub)sample mean as a function of the (sub)sample size can also be used to distinguish between short memory processes, long memory processes and "over differenced" processes.

Given the correlation function in (6) one can easily generalize the results for $\widehat{\beta}_0$ to $\widehat{\beta}_1$, $\widehat{\beta}_2$, and $\widehat{\beta}_3$. For short memory processes one has $\mathrm{var}\left(\widehat{\beta}_1\right) \sim 2\pi T^{\ \ 1} f_y(\pi)$, $\mathrm{var}\left(\widehat{\beta}_2\right) = \mathrm{var}\left(\widehat{\beta}_3\right) \sim 4\pi T^{\ \ 1} f_y(\pi/2)$, $T \to \infty$, see Anderson (1971, Th. 8.3.8). For the process $y_t = \beta_1(\ \ 1)^t + (1 + L)^{\ \ d_{1/2}} u_t$, with u_t white noise one gets

$$var\left(\widehat{\beta}_1\right) \sim \sigma_y^2 T^{(2d_{1/2})\ \ 1}, T \to \infty$$

for the process $y_t = \beta_2 \cos(\pi t/2) + \beta_3 \sin(\pi t/2) + \ \ 1 + L^2)^{\ \ d_{1/4}} u_t$ one obtains

$$var\left(\widehat{\beta}_2\right) = var\left(\widehat{\beta}_3\right) \sim \sigma_y^2 (T/2)^{(2d_{1/4})\ \ 1}, T \to \infty.$$

The faster rate of increase of the spectrum near the singularity at the yearly frequency corresponds to a slower increase in the variance time function, compared with the corresponding model with zero frequency integration. These expressions are the *flexible seasonal* analogs of the *variance-time function*. Adenstedt (1974, Remark 9) discussed the behavior of the asymptotically best linear unbiased estimator of β_0, which is slightly more efficient than $\widehat{\beta}_0$. He also showed how to generalize the results to processes with long memory behavior at any particular frequency, so as to derive BLUE estimators for β_1, β_2 and β_3 in those cases. According to his results the OLS estimator using (16) is asymptotically not very inefficient for positive orders of integration. Of course, standard errors for the parameter estimates for the deterministic part depend crucially on the orders of integration at the frequencies of interest.

It is clear that the periodogram ordinates $I_T(2\pi j/s)$ at frequencies j/s with long memory do not behave as in the short memory case. Their means and variances are time dependent, even asymptotically. For negative orders of integration, their variances tends to zero. There is good reason to omit these ordinates in the estimation process.

The special properties of $\overline{y}_T$ for the process $y_t = \beta_0 + (1 \quad L)^{\quad d_0} u_t$ have repercussions on the corresponding *separate sample covariances* $\widehat{\gamma}_k = T^{\quad 1} \sum (y_t \quad \overline{y}_T)(y_{t \quad k} \quad \overline{y}_T)$, which converge slowly to their population values for $d_0 \geq 0.25$. The same holds for the sample autocorrelations, see e.g. Newbold and Agiakloglou (1993). Those are also extremely biased towards zero for high values of d_0 even in moderately sized samples (bias about -0.1 for ρ_1 if $d = 1/3$ and $T = 200$). If the process is integrated at frequency $\frac{1}{2}$ or $\frac{1}{4}$ one expects similar problems for the separate sample covariances adjusted for seasonal means: $\widehat{\gamma}_k' = T^{\quad 1} \sum \widehat{x}_t \widehat{x}_{t \quad k}$ where $\widehat{x}_t$ is the estimated residual from (16).

Sowell's (1992b) ML estimator requires estimation of the mean, and inherits some of its problems in finite samples. The frequency domain estimator presented above does not have this problem, since it does not depend on $\widehat{\beta}_0$, $\widehat{\beta}_1$, $\widehat{\beta}_2$, and $\widehat{\beta}_3$, from regression (16). One should note that *individual* periodogram ordinates at the Fourier frequencies which tend to the singular points have unusual limiting distributions as well, see e.g. Künsch (1986).

In order to get reliable estimates of the correlation structure and corresponding impulse responses of the process one has to estimate a model (preferably with fully specified spectrum) and avoid using too few covariances in the time domain, or using too few periodogram points in the frequency domain. Model selection can follow the usual paths, inspection of diagnostics, or maximization with respect to likelihood based information criteria.

Finite sample improvements, model selection and robustification

It is not certain which estimation method is preferable in finite samples. In order to distinguish between short memory ARMA models with roots close to the unit circle and truly long memory models one will always need many observations anyway, see e.g. Robinson (1994) on power for frequency domain LM tests and Agiakloglou and Newbold (1994) for time domain LM tests.

For the nonseasonal ARFIMA model one has developed approximating formulae for finite sample biases in the estimated autocorrelations, which can be used to improve the precision of e.g. GMM estimators in simple cases, see Mikkelsen (1994). In frequency domain estimation one has suggested the use of several *data tapers*, which downweight the effects of observations towards the beginning and towards the end of the sample as a bias reduction device. The use of tapers on economic time series seems to be rare.

Gaussianity is not essential asymptotically, since a lot of the asymptotic theory carries over for other finite variance distributions for the innovations. One can use the AR approximations (2) (4) of the model to apply data cleaning in order to avoid problems with large outliers. Ooms (1994) applied a so-called robust filtering method, but found outliers not to matter too much in his case. Martin and Yohai (1991) provided an overview of the main existing techniques for robust estimation of AR models, which could also be applied directly. Beran (1994) examined several robust M(aximum likelihood–type) estimators to estimate long memory models.

(Seasonal) Unit root tests

Testing for (seasonal) unit roots, i.e. testing for the orders of integration $d_{j/s}$, is important in the identification stage of the modeling process. Notice that (optimality) properties of estimators are derived under the assumption that $0.5 \leq d_{j/s} < 0.5$. In practice one often does not know a priori whether the actual orders are in this region, and tests can help to decide the appropriate transformation of the process to get it into this region for all $d_{j/s}$.

Robinson (1994) used a general form of model (1) to develop LM tests for the orders of integration $d_{j/s}$ at frequency j/s against fractional alternatives $d_{j/s} + v$, with ν a fixed real number unequal to zero. These tests differ with "integer unit root tests" which only allow for integer ds under the null and the alternative. Using efficient estimates of the unrestricted model one can apply Wald tests, with usual χ^2 based asymptotic critical regions. These tests are likely to be more efficient than LM tests farther away from the null.

Integer unit root tests of integration $d_{j/s} = 1$ against $d_{j/s} = 0$ involve testing $\phi(e^{i2\pi j/s}) = 0$ against $\phi(e^{i2\pi j/s}) \neq 0$ and vice versa. Ooms (1994, pp. 37-58) provides an extensive discussion. For $d_{1/4}$ and $p = 2$ this amounts e.g. to testing the null

$$H_0 : \phi(z) = (1 + z^2) \text{ against } \quad H_1 : \phi(z) = (1 + \rho^2 z^2),\ 0 < \rho < 1. \tag{17}$$

so that $\phi(z)$ becomes part of $D(z)$ under H_0.

Integer unit root tests are needed to evaluate the estimates of the short memory part of the model. It is important to check whether the roots of the estimated "short memory" part of the model are well inside the stationary region. We apply a test based on the estimated roots of the AR part of the model. In the context of AR(p) models $\phi(L)x_t = \epsilon_t$, Fountis and Dickey (1989) derived a distribution of $T(1 \quad \widehat{\lambda}_p)$ with $\widehat{\lambda}_p$ the root of $\widehat{\phi}(z) = 0$ closest to 1, to test $\lambda_p = 1$ against $|\lambda_p| < 1$, with finite p and all other roots inside the stationarity region. Ooms (1994) found a modification of this test useful for seasonal unit roots as well. The corresponding Dickey-Fuller distributions do not apply in the event of truly fractional $d_{j/s}$ in the long memory part of the model, see Sowell (1990)), but they may give a useful indication of the significance of the distance of the root to the unit circle.

4 Illustration of periodogram regression

In this section we present a small Monte Carlo study which shows that the asymptotic theory on the most straightforward linear estimator of fractional integration parameters can be applied directly to a flexible seasonal long memory model. We employ this study also to spot the differences between the seasonal periodogram regression procedures of Hassler (1994) and Ooms (1994), with respect to estimation and (flexible seasonal) unit root testing.

Tests based on periodogram regression (see e.g. Hassler (1993), Cheung (1993), Hurvich and Ray (1995)), are asymptotically less powerful than tests based on the approximate ML estimators discussed in the previous section. There is a substantial difference in the asymptotic efficiency of these two types of estimators: In a fractional ARIMA$(0,d,0)_1$ model the Cramer-Rao bound on the asymptotic variance of $\sqrt{T}(\widehat{d_0} \quad d_0)$, which applies to the ML estimators, is $6/\pi^2$, see Ray(1991, Th. 4.2). This is about 60% of the optimal variance of periodogram regressor estimators which have minimal asymptotic variance of approximately $1/T$. However, this does not make periodogram regression estimators useless. One can make periodogram tests less dependent on the specification of the high frequency part of the model. They are robust in other ways as well, see Cheung (1993). The efficiency of periodogram tests can be improved by periodogram smoothing, see Boes et al. (1989), Beran (1992). In sum, variants of periodogram estimators may be preferred over approximate ML estimators in terms of robustness and in terms of finite sample performance.

In this section we examine the model

$$(1 + L^2)^{d_{1/4}} x_t = \epsilon_t \tag{18}$$

Since there is only one $d_{j/s}$ in this model, we drop the subscript 1/4 for the rest of this section: $d = d_{1/4}$.

Data generating process

We generate data from model (18) with ϵ_t Gaussian white noise. We employ the exact autocovariance function (6) for series x of lengths $T = 100$, 200, and 400. We compute the triangular Choleski factor $A(d)$ such that $A(d)' A(d) = Q(d)$ in (10). Next we compute $A(d)\, u$, with u a vector of white noise of corresponding length. We use the random generator from GAUSS 3.2.4 on a SUN Work station. We vary d from 0.99 to 0.49. We present results for representative cases below. For $d_{1/4} <$ 0.5 we generate series of length $T + 1$ using $A(d + 1)$ and take differences. Direct application of $A(d)u_t$ gives the same results. We use 2000 replications, which entails a maximum standard error in the empirical distribution function of about 0.01 near the median, and a standard error of about 0.005 near the 0.05 and 0.95 quantile.

Periodogram Estimators

We employ two simple least squares estimators of $d_{1/4.}$ We regress the log periodogram, $I_T(2\pi i/T)$ see (12), on a constant and one regressor. We vary the regression range. The spectral regressor employed by Ooms(1994) reads (simply using the log spectrum (5) substituting $j/s = 1/4$):

$$X_i = \quad 2 \ln |2 \cos (2\pi i/T)| \tag{19}$$

Hassler(1994, eq. (3.1)) used

$$R_i = \quad \ln \left(4 \sin^2 (\pi i/T \quad \pi/4)\right) \tag{20}$$

where $|i \quad T/4| = 1, 2, \ldots [n/2]$. n denotes the number of periodogram points used. For fixed n and under the assumption of Gaussian innovations Hassler indicated that one can apply the ordinary least squares formulae to derive asymptotic standard errors. The asymptotic regression error variance is known and equals $\pi^2/6$. So we apply the regressions

$$I_T(\omega_i) = c_X + d_X X_{(\omega_i)} + u_{X,i} \tag{21}$$

and

$$I_T(\omega_i) = c_R + d_R R_{(\omega_i)} + u_{R,i} \tag{22}$$

where c_X and c_R denote constants, d_X and d_R denote the integration parameter to be estimated, $u_{X,i}$ and $u_{R,i}$ denote error terms, and ω_i denotes the Fourier frequencies used in estimation: $|\omega_i \quad \pi/2| = \frac{2\pi}{T}, \frac{4\pi}{T}, \ldots, \frac{n\pi}{T}$ .The regression (22) provides slightly (also asymptotically) biased estimates if $d \neq 0$. Hassler gave formulae to approximate the bias. As is clear from figure (1) the regressors (19) and (20) are very similar around $\omega = \pi/2$. As long as n is small, the differences between regressions (21) and (22) are negligible.

Suppose we wanted to apply (21) to estimate $d_{1/4}$ for a general process with an additional long memory part and probably an extended short memory part. One can view this as an omitted regressor problem. It is clear from Figure 1 that the regressors for the estimation of $d_{j/s}$, $j/s \neq 1/4$, are not orthogonal to the regressors in (21) if one gets farther away from $\omega = \pi/2$. Therefore, omitted variable bias results if other long memory components are neglected. Multiple regression is a better option in that case. Similar bias results for neglected short memory parts of the process, but this bias disappears asymptotically if the number of observations T grows much faster than the number of periodogram points n.

$$Insert\ Figures\ 3\ ,\ 4\ \ 5\ and\ 6$$

Results

We present the simulation results in plots of empirical distribution functions in figures 3, 4, 5 and 6. The figures show the empirical CDFs of the OLS estimators obtained by regressing the $n = [T/2] \quad 2$ periodogram points on a constant and the regressors in (19) and (20), respectively, for $T = 100, 200$, and 400. Figures 4, 5 present the results for $T = 200$ for two values of n : 98 and 24. We plot the asymptotic Gaussian CDF based on ordinary regression theory (with known error variance and regressor (19)) in the same figures. The variance of the asymptotic CDF does not depend on d in the region $(\quad 0.5, 0.5)$.

The value for d of $\quad 0.8$ is outside the region where the asymptotic theory is believed to apply, see Hurvich and Ray (1995) for an analysis of periodogram regressor for values outside the region $(\quad 0.5, 0.5)$. The other values of d for which we present estimation results are inside the region $(\quad 0.5, 0.5)$.

The figures show that periodogram estimator (19) works better than (20), for $n = T/2$. Absolute median bias is somewhat smaller, especially for negative d. The estimator (21) shows a positive bias, which is only significant for negative d, whereas (22) has a positive bias for negative d and a negative bias for positive d. Both estimators are severely biased for $d = \quad 0.8$.

$$Insert\_Table\_1\ \ here$$

Table 1 contains results on the bias and root mean squared error (RMSE) for more values of d. For $n = [T/2] \quad 2$ we report results for both estimators. For $T = 200$ and $n = 24$, we report only results for one estimator, since the differences between the two estimators are negligible in that case, see also the right hand side plots in Figures 4, 5.

Columns (1) , (5) and (11) in Table 1 show that the bias for the R estimator from (22) is negatively related to d. Hassler (1994, p. 26) provided a formula for this bias. The bias for the X estimator from (21) is uniformly smaller in absolute value, especially for very negative and very positive values of d, see columns (2), (6) and (12) . However, the variance for the R estimator is smaller than for the X estimator, leading to a smaller RMSE for positive values of d, for $T = 100$ and $T = 200$. This advantage of the R estimator decreases with sample size.

Figure (6) presents the results for the largest sample of our analysis. The theoretical standard deviation of regression estimators of d using the maximum number of (approximately) independent periodogram points is $T^{\ \ 1/2}$, see Ray (1991), which is 0.05 for $T = 400$, cf. column (14) in Table 1. The theoretical standard deviation provides a good approximation in this case.

Let us look at the effect of reducing the number of periodogram points, cf. columns (6) and (8) with columns (9) and (10). This procedure has been suggested to make the regression applicable for direct estimation of d in more general situations. The basic idea is then that the spectrum of the weakly dependent components in that smaller region can be considered constant for practical purposes, so that omitted variable bias in the regression can be neglected. It is also suggested to leave out a couple of periodogram points very close to the long memory frequency (see Hurvich and Ray(1995)), because asymptotic distribution theory for these points (mean, convergence rate and independence) is somewhat different, but this seems to be of minor practical importance for sample sizes considered here.

The bias for the X estimator seems to increase by using fewer periodogram points, cf. columns (6) and (9). Naturally, the RMSE increases substantially. The RMSE for $n = 24$ and $T = 200$, 0.18, is larger than the RMSE for $n = 48$, and $T = 100$, 0.13. The theoretical variances seem to give a good indication of the finite sample variation, see Figures 3, 4, and 5. This suggests that reliable (unit root) tests on the value of d can be performed using regression (21). We examine this more closely in Table 2.

Insert Table 2

In Table 2 we report the empirical critical values of the normal CDF-value of the standardized estimate of d. Asymptotically this CDF-value (which is easily computed by most standard packages which are able to present p-values for standard test statistics) has a uniform distribution on the unit interval. The theoretical critical value for a 5% test against smaller alternatives is 0.05. The critical value for a 5% test against larger alternatives is 0.95. All in all the test procedure seems to work well for nearly the whole region $d \in (\ \ 0.5, 0.5)$. The empirical distribution of the test statistic based on X in (21) is somewhat shifted and skewed to the right for large positive and negative values of d, and somewhat shifted to the left for absolutely small values of d. This difference is hardly significant. The test based on R in (22) is adversely affected by the estimation bias. It rejects null hypotheses too infrequently against smaller alternatives for d if d is small. It rejects too infrequently against larger alternatives for d if d is large, see e.g. columns (11) and (13). The null of $d = \ \ 0.8$ is rejected too infrequently against smaller alternatives and too often against larger alternatives. The test for $d = 0.2$ is hardly biased, so it is better not to "over difference" before doing these unit root tests.

Insert_Table_3

Since these tests are reasonably reliable in size, it makes sense to look at the power function. Table 3 contains information on so-called "size-adjusted" power, i.e. power computed using empirical critical values, rather that theoretical critical values. The table reports power of the test for $d = 0$ only. The differences between the two tests are negligible for this value of d. Power against positive values of d seems to be somewhat higher than the power against the corresponding negative values of d. Reducing the number of periodogram points, cf. columns (8) and (10), decreases power substantially. Suppose one has 50 years of quarterly data, and $d = 0.3$, a realistic value for inflation rates. Then power of a test against $d = 0$, reduces from 0.98 to 0.54 by reducing the number of periodogram points from 98 to 24. The price one has to pay for increased robustness is substantial in this case.

Concluding remarks on periodogram regression

Summarizing the results one sees that simple periodogram regression on the theoretical spectrum works fine in the simple seasonal long memory model considered. Reasonably efficient statistical inference is possible if one uses sufficiently many periodogram points. The Monte Carlo results for model (18) with periodogram regression (21) correspond closely to the results obtained by other authors for the standard model $x_t = (1 \quad L)^{\quad d_0}$, using the standard periodogram regression and similar numbers of observations and periodogram points.

Economic time series often exhibit more complicated correlation functions. Generalization of the technique to multiple regression on more long memory components seems straightforward. Simultaneous introduction of regressors for weakly dependent correlation does not lend itself to tractable statistical analysis, so that one has to apply nonlinear estimators in that case as introduced above. These alternative estimators are more efficient, at least asymptotically. Finite sample analysis is hardly available, but since long memory is not a particularly interesting subject to examine in short time series, where usual ARMA models can approximate the long memory behavior to a great extent without using too many parameters, this may not be such a problem in practice.

5 US GNP

In this section we apply approximate frequency domain ML (FDML) and exact ML (EML) to the series of quarterly changes in the seasonally adjusted series for the logarithm of US GNP (1947.2-1989.4). First we apply the ARIMA $(3, d, 2)$ model selected by Sowell (1992a). Next we extend the analysis to include flexible seasonal long memory.

Comparing FDML and EML

We redo the exact ML estimation using Sowell's algorithm to evaluate the likelihood and standard errors of the estimators. We compare it with results from FDML estimation using (14). Partial plots (not reproduced) show that the log likelihood behaves well in a (two standard error) neighborhood of the parameter estimates. Only the second MA parameter may be too close to 1 for comfort. The parameter estimates are in table 4. The estimates for d_0 present the order of integration for the growth rates. The familiar integer unit root hypothesis for the log levels corresponds to $d_0 = 0$ in this table.

Insert Table 4

The most striking difference between the estimation results in rows Ia, Ib, IIa and IIb is in the size of the approximate standard error of $\widehat{d_0}$ which is larger for EML. These standard errors are based on numerical approximation of the Hessian of the log likelihood in both cases. The bootstrap standard deviations, computed by Sowell (1992a) for the EML parameters, indicate that the reported EML asymptotic standard error is too large in this case, see row IIc. The estimate of the standard error of the white noise residuals is also larger using EML. The estimate of d_0 is about 0.1 lower using FDML, but the basic finding that statistical analysis cannot decide whether log US GNP is trend stationary or not remains valid. Given the assumption that ML estimation and inference works for the region $d \in (\quad 1, \quad 0.5)$, as the simulation exercise by Sowell suggests, we get a 95% confidence interval for the order of integration of the log *level* of about $(\quad 0.2, 0.8)$, which includes both stationary $(d < 0.5)$ and nonstationary $(d > 0.5)$ values.

$Insert Figure\, 7$

The ARMA parameters are not easy to interpret as such. Therefore we present the periodogram and the spectrum of the two estimated models in figure 7. This shows the estimated model spectra to be very close indeed.

Changing the ARMA structure has a bigger influence on the (conditional) inference on d than changing the approximation of the likelihood function, compare the outcome for the ARIMA(1,d,0) model in rows IIIa and IIIb in the table 4 with rows IIa and IIb. The estimates for this alternative model, which Sowell selected using the Schwarz information criterion, suggests the process is closer to having an integer unit root in the levels if one looks at d_0: -.45 for ARIMA(1,d,0) versus -0.59 for ARIMA(3,d,2). The dominant root in the AR part is farther away from 1, however: 0.77 versus 0.81.

Seasonal adjustment and seasonal overdifferencing

The US data are seasonally adjusted. It is preferable to use unadjusted data but these are not always available for aggregated data. Commonly used moving average filters for seasonal adjustment, like the one implied by the linear part of the quarterly Census X-11-procedure, have a double unit root at the seasonal frequencies, see Laroque (1977, Tableau 3), which, given the rarity of series integrated of order 2 at the seasonal frequencies, is very likely to induce a negative order of integration for the seasonally adjusted series. Here the seasonal adjustment leads to dips in the spectrum at the seasonal frequencies 1/4 and 1/2. See Bell and Hillmer (1984) for an explanation why these "spectral dips" arise naturally in the context of an unobserved components representation and estimation of seasonal ARIMA models. One would expect the estimated model spectrum to follow these dips in order decrease the residual variance term in the log likelihood.

Extension with extra MA parameters was not successful in our "reduced form" approach in modeling the dips. Direct application of the flexible seasonal long memory model is, as is shown the last rows of table 4. In order to apply the formulae directly to the standard Fourier frequencies $2\pi i/T$ of the periodogram, the series is truncated to have only full years. We lose 3 early observations in this case. The truncation has the big advantage that the FDML estimator does not change by prior regression of the growth rates on a constant and seasonal dummies.

The flexible $\mathrm{ARIMA}(1, d, 0)_s$ model turns out to be an improvement on Sowell's preferred $\mathrm{ARIMA}(3, d, 2)$ model, achieving a lower residual variance with a smaller number of parameters. What exactly is the relevant number of parameters is debatable. It depends on whether the exclusion of periodogram points necessary in the estimation in the flexible model is regarded as the estimation of an extra number of parameters or not. In the former case the models have 9 $(\beta_0, \ldots, \beta_3,\ d_0, d_{1/4}, d_{1/2}, \phi_1, \sigma_e^2)$ and 8 $(\beta_0,\ d_0, \phi_1, \phi_2, \phi_3, \theta_1, \theta_2, \sigma_e^2)$ parameters respectively, in the latter case 5 and 7. Given the limited information in these periodogram points, we support the latter interpretation in the context of model selection for this seasonally adjusted series. The implied spectra of both models are plotted in Figure 8. The spectral shapes of these models differ considerably, especially at the higher frequencies.

$Insert\ Figure\ 8$

In the next section we apply the flexible seasonal long memory model to unadjusted data of considerable length, which makes the difference between the long memory part and the weakly dependent part come out more clearly.

6 Shipping through Sound

In this section we examine the number of ships passing through the Sound (eastward and westward), registered by Danish customs at the renowned castle of Kronborg near Elsinore. The series is described (in extreme detail) in Bang (1906) and Bang and Korst (1923). We use monthly data from 1557.1 to 1783.12. We show time series plots in figure 9. The lower panel shows only the last years in order to bring out the relatively smooth seasonal intra-year pattern with an important frequency $\frac{1}{12}$ component.

Insert Figure 9

Data and historical perspective

The Sound was (and is) one of the most important shipping lanes in Europe. It forms the bottleneck in the connection between the North Sea in the west and the Baltic in the east. The series is influenced by a number of processes which possibly are long memory. One can think about economic activity depending on persistent price differences for tradable goods (manufacturing products, food, wood) between eastern and western Europe. Climatic changes may have had long lasting effects on the seasonal pattern in trade, especially in the times when technological change was not able to accommodate these changes. Political changes, leading to long periods of war and peace and to persistent changes in the tariff structure were probably most influential. We discuss some effects below. Economic reasons can be considered to be important behind these changes as well. Technological changes in combination with the way in which tariffs were determined had their long lasting influence on the average size of the ships.

The time series contains some gaps at times of war in the area. The first ones occur for the years 1559, 1561 and 1570-1573, in and around the seven year war between Denmark and Sweden (1563-1570). Although we know the Sound to have been partly closed during these periods, we treat these observations, statistically speaking, as missing. We backcast values for these years using a flexible periodic AR(12) model in VAR form (see Franses(1994)), to fill up the gaps and take absolute values. We forecast the values for 1632 and 1634, and for the years 1658-1660. The latter years are missing because of Swedish occupation of Kronborg. Values after 1783 do not exist, because the (new) Danish government adopted free trade policies in 1784, which ended the precise registration of the passing ships and their cargoes.

Taking a closer look at figure 9 we see that the series contains some very long non-periodic cycles, characteristic of long memory processes. These cycles correspond to the rise and fall of important (shipping) nations. Dutch interests were paramount in the first century of this period. The first century long cycle (up to 1658) corresponds to the rise and fall of Dutch shipping, which made up 80 to 90% of the observed number of ships at times. The predominant influence of Sweden in the area rose and fell in the following period up to 1720. The first 20 years of the 18th century saw many wars in the Baltic area, with a steady decline of Swedish influence. The continuous upsurge from 1720 on reflects an unprecedented (and unsurpassed) long period of peace in the area, which lasted until 1807, when war broke out again when Britain defended its economic interests against continental Napoleonic Europe.

Some of the troughs in the overall pattern (1658, 1709) correspond to ("little ice age-like") periods with very cold winters where the Sound (and the major part of the Baltic) froze for long periods (no ships until April). This indicates that climatic changes naturally had a marked influence on trade. Overall economic (causes and/or) consequences of war and peace seem to be the dominant long run factors, though.

Insert_Figure 10

Time series analysis

The sample autocorrelations in figure 10 clearly represent long memory characteristics already observed in the time series plot. The correlations for the levels, also those corrected for trends with a monthly varying slope, die out very slowly. The correlations of the first differences show a comparatively smooth intra year pattern (with dominating oscillating factor $\cos k\pi/6$, with k the lag in months) which dies out very slowly. This indicates dominant long run behavior of the frequency 1/12 component. The correlations for the yearly changes are predominantly negative for the first 5 years and approach zero faster. This is a sign of "overdifferencing", "anti-persistence", or a negative order of integration.

We estimated an flexible seasonal ARIMA$(12, d, 0)_{12}$ model for $(1 \quad L^{12})x_t$ and $(1 \quad L)x_t$, both for the period 1558.1 1783.12, i.e. 2712 observations. The winter months sometimes have zero values, so that a log transformation is not possible. The model for the yearly changes was inferior to the model in monthly changes. Its FDML estimates of $d_{j/12}$ were all in the region $(\quad 0.1, \quad 0.5)$ for $j > 0$. The model in first differences produced more satisfactory results which we present in table 5. In the same table we present the results of two periodogram estimators. The first one uses all periodogram points except those at the seasonal frequencies and their immediate neighbors. The next one effectively employs 55 points or 54 (i.e. about $T^{1/2}$) per integration parameter, following common practical guidelines for the use of periodogram regression estimators.

Insert Table 5.

We set the AR order of the model at 12 to allow for all integer seasonal unit roots as well, although not all parameters are separately significant from zero. The results do not indicate integer unit roots. All roots of the AR part, i.e. the roots of $\phi(z) = 0$, are well inside the stationarity region. The fractional integration parameters are all in the preferred region for statistical inference, $(\quad 0.5, 0.5)$, although d_0 and $d_{1/12}$ are not significantly away from the borders of this interval. These results entail that the series is definitely "mean" stationary at the frequencies greater than 1/12. The zero frequency order of integration for the levels, $d_0 + 1$ is 0.6, $d_{1/12}$ equals about 0.5. The orders of integration at the other frequencies $j/12$ decrease monotonically with j, and are not significantly different from zero for frequencies 5/12 and 1/2.

The periodogram estimates in the last columns of Table 5 are difficult to interpret given that the weakly dependent part of the model differs significantly from white noise. Nevertheless, they seem to confirm the previous conclusions.

How do we interpret these outcomes? Remember first that the estimates do not depend on the sample mean growth rates for the different months. The model picks up deviations from these means (which exist if $d_{j/12} < 0.5\ \forall j$). We present OLS estimates for these means in Table 5. Long run changes in the seasonal pattern seem to move with the yearly mean of the data, since winter figures hardly move in comparison with the data for the other months. This entails that the pattern at the yearly frequency is as much affected by long run changes as the zero frequency aspect of the series. Technology shocks allowing substantial increase in long distance shipping in the cold, dark and windy winter months occurred only long after the sample period.

Insert Figure_11

Insert Figure 12

We evaluate model adequacy primarily in the frequency domain. In figure 11 we present the periodogram of the monthly changes and the spectrum of the model. In Figure 12 we plot the frequency domain residuals $I(\omega_i)/\left(g_x\left(\omega_i,\widetilde{\psi}\right)\right)$, (with $g_x\left(\omega_i,\widetilde{\psi}\right)$ defined under (14) or *residual periodogram*, representing the spectral characteristics of the supposedly white noise innovations. We investigate the "whiteness" of these residuals further by inspection of a *centered cumulative periodogram* plot, using the idea from Durbin (1969), who analyzed the periodogram of regression residuals. Since the residual periodogram points should be approximately i.i.d. (except a few ones very close to the long memory frequencies),

$$D_l = \frac{T^{1/2}}{2}\left[\sum_{i=1}^{l} I(\omega_i)/\left(g_x\left(\omega_i,\widetilde{\psi}\right)\right)\right] / \left[\sum_{i=1}^{[T/2]} I(\omega_i)/\left(g_x\left(\omega_i,\widetilde{\psi}\right)\right)\right] \quad l/\left[T/2\right],\ l=1,\ldots,[T/2] \tag{23}$$

should behave approximately as a (discrete version) of a tied down Brownian motion, for which asymptotic critical regions for minimum, maximum and range have been derived. The asymptotic 5% critical value for $\max_l |D_l|$ is approximately 1.36. Inclán and Tiao (1994) suggested to subtract the last term in this type of cumulative plot in the context of testing for homoskedasticity. They called it a *centered cumulative sums of squares plot*, and labeled it D_k. In Figure 13 we present the 5% critical values for the maximum and the minimum of $2D_l/T^{1/2}$ which we plot for all l to get an indication of gross errors in dynamic specification in the model.

Insert_Figure_13

Insert_Figure_14

Since we are using residuals from a dynamic model and 7 of the 1356 residual periodogram points are put to zero, these critical values are probably a bit conservative. Values below zero in these plots indicate dominance of high frequency components over low frequency components, values above zero entail the opposite. Here the plot does not indicate rejection of a white noise error term. Figure 14 plots the cumulative expected values for the periodogram predicted by the model. It also presents contributions of the long memory part and the AR part separately. The AR part is seen to pick up high frequency components. The spectrum for the AR part is only approximately constant in the neighborhood of frequency $\frac{1}{3}$. Therefore the different estimates of $d_{1/3}$ around 0.22 in Table 5 are pretty similar. The spectrum of the AR part is far from constant near the frequency $\frac{1}{2}$, see also Figure 15. The long memory (FI) part leads to "steps" in the cumulative periodogram around seasonal frequencies with positive orders of integration. The low frequency parts of the AR and FI model components resemble each other.

Insert_Figure_15

Insert_Figure_16

Finally we present the adequacy of the model for the *log* periodogram in Figure 16, where the negative order of integration for the growth rate at frequency zero comes out more clearly.

We have not looked into time domain characteristics, which may indicate shortcomings of the model, like seasonal heteroskedasticity, fat tailed innovation distributions etc. The frequency domain characteristics of the unadjusted series seem to be well captured though.

7 Conclusion

Flexible frequency domain modeling of seasonal long memory in economic time series is useful, and does not require excessive computational effort. It should also be considered for seasonally adjusted data if unadjusted data are unavailable.

Reliable frequency domain estimation of simple flexible seasonal long memory models is possible in moderately sized samples, provided one uses sufficiently many periodogram points and a fully specified model. Regression procedures work well in some cases. We successfully modified the periodogram regression procedure of Hassler (1994).

We compare exact (Gaussian) ML (EML) and approximate frequency domain ML (FDML) estimates of a fractional ARFIMA model for US GNP suggested by Sowell (1992a), observing smaller approximate standard errors for the fractional integration parameters, but similar spectral characteristics of the estimated model. We successfully apply a flexible seasonal fractional ARIMA model, which proves to be more adequate for these data.

We apply a flexible seasonal fraction ARFIMA model to monthly shipping data for the Sound (1557-1783). We find evidence of fractional orders of integration with orders of integration varying significantly from frequency to frequency, with orders of integration of about 0.5 for the zero and the yearly frequency.

Suggestions for further research are: evaluation of the nonlinear estimation procedures in finite samples, where proper estimation of standard errors of parameters seems to be problematic, development and evaluation of more time domain diagnostics, and comparison with periodic models. The empirical results can be evaluated using other estimators and other models generating such autocorrelations patterns, such as models with infrequent structural breaks.

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Table 1: *Bias and RMSE $\widehat{d}_{1/4}$ in periodogram regressions*

	$T = 100$				$T = 200$						$T = 400$			
	$n = 48$				$n = 98$				$n = 24$		$n = 198$			
	bias		RMSE		bias		RMSE		bias	RMS	bias		RMSE	
no.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
d^*	R	X	R	X	R	X	R	X	X	X	R	X	R	X
-.800	.157	.090	.202	.168	.139	.077	.167	.127	.170	.271	.121	.061	.137	.094
-.500	.078	.033	.145	.139	.067	.025	.105	.092	.045	.196	.057	.016	.077	.059
-.450	.064	.023	.137	.136	.057	.019	.096	.087	.041	.191	.050	.014	.072	.058
-.400	.056	.019	.133	.134	.046	.012	.091	.086	.028	.191	.039	.006	.064	.055
-.300	.040	.011	.127	.135	.032	.006	.081	.083	.011	.184	.029	.005	.059	.056
-.250	.031	.008	.121	.130	.027	.005	.082	.085	.013	.179	.024	.003	.055	.054
-.200	.021	.002	.119	.129	.018	-.000	.078	.083	.002	.182	.016	-.000	.053	.055
-.100	.016	.007	.115	.126	.010	.001	.078	.085	.003	.181	.007	-.002	.051	.055
.000	-.002	-.002	.117	.129	.001	.001	.077	.085	.001	.182	-.000	-.000	.050	.054
.100	-.010	-.001	.120	.131	-.008	.000	.078	.085	-.001	.185	-.008	.000	.051	.055
.200	-.018	.001	.118	.130	-.016	.002	.077	.082	.003	.181	-.019	-.002	.055	.056
.250	-.018	.007	.119	.130	-.019	.003	.078	.082	-.000	.179	-.021	-.000	.055	.055
.300	-.019	.011	.119	.130	-.022	.005	.078	.082	.002	.182	-.022	.004	.055	.055
.400	-.025	.016	.118	.128	-.028	.009	.080	.083	.017	.184	-.029	.006	.057	.055
.450	-.033	.013	.121	.128	-.028	.014	.082	.086	.020	.184	-.029	.010	.057	.054
.490	-.033	.017	.121	.130	-.027	.018	.082	.087	.026	.182	-.032	.011	.060	.057

NOTE: T: no. of observations, n: no. of periodogram points. d^*: order of integration at frequency 1/4. Regressors: R in (20), X in (19). Results based on 2000 independent replications for each T and d^*. Results for R for $T = 200$, $n = 24$ (not reported) very close to those for X.

Table 2: *Empirical critical values of test statistic for $d_{1/4} = d^*$*

	$T=100$				$T=200$						$T=400$			
	$n=48$				$n=98$				$n=24$		$n=198$			
	.050		.950		.050		.950		.050	.950	.050		.950	
no.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
d^*	R	X	R	X	R	X	R	X	X	X	R	X	R	X
-.800	.308	.129	.999	.993	.401	.108	1.000	.998	.148	.998	.655	.182	1.000	1.000
-.500	.129	.060	.990	.970	.174	.063	.995	.976	.058	.973	.268	.074	.998	.978
-.450	.114	.056	.987	.967	.165	.065	.992	.970	.059	.972	.239	.069	.996	.969
-.400	.095	.049	.984	.966	.122	.050	.989	.964	.048	.965	.178	.056	.990	.955
-.300	.076	.042	.976	.959	.108	.056	.978	.953	.043	.948	.134	.055	.985	.955
-.250	.085	.055	.969	.950	.079	.041	.976	.952	.055	.955	.113	.050	.982	.951
-.200	.060	.045	.966	.950	.071	.044	.968	.947	.038	.945	.086	.043	.976	.948
-.100	.068	.057	.952	.944	.051	.038	.961	.950	.049	.949	.057	.041	.957	.939
.000	.038	.037	.944	.943	.046	.046	.952	.954	.038	.946	.044	.044	.938	.937
.100	.030	.035	.944	.948	.027	.036	.927	.941	.036	.947	.034	.048	.927	.944
.200	.031	.043	.923	.943	.029	.048	.922	.949	.044	.943	.015	.031	.888	.939
.250	.030	.045	.924	.950	.025	.046	.907	.945	.045	.944	.019	.050	.887	.948
.300	.028	.050	.929	.956	.024	.055	.901	.948	.044	.951	.016	.051	.882	.953
.400	.028	.054	.910	.952	.018	.051	.885	.954	.055	.957	.014	.064	.851	.955
.450	.022	.050	.899	.949	.017	.060	.885	.961	.052	.958	.012	.065	.833	.956
.490	.027	.061	.907	.960	.016	.059	.895	.964	.059	.959	.009	.063	.850	.967

NOTE: see Table 1. Test statistic: $\Phi((\widehat{d}_{1/4} \quad d^*)/\sigma_{\widehat{d}})$, with $\Phi(.)$ the standard Gaussian CDF.

Table 3: *Powers of one-sided test of $d_{1/4} = 0$ at 5% level*

	$T=100$				$T=200$						$T=400$			
	$n=48$				$n=98$				$n=24$		$n=198$			
	$<$		$>$		$<$		$>$		$<$	$>$	$<$		$>$	
no.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
d^*	R	X	R	X	R	X	R	X	X	X	R	X	R	X
-0.80	1.00	1.00	0.00	0.00	1.00	1.00	0.00	0.00	0.94	0.00	1.00	1.00	0.00	0.00
-0.50	0.97	0.97	0.00	0.00	1.00	1.00	0.00	0.00	0.76	0.00	1.00	1.00	0.00	0.00
-0.45	0.94	0.94	0.00	0.00	1.00	1.00	0.00	0.00	0.69	0.00	1.00	1.00	0.00	0.00
-0.40	0.88	0.88	0.00	0.00	1.00	1.00	0.00	0.00	0.59	0.00	1.00	1.00	0.00	0.00
-0.30	0.67	0.66	0.00	0.00	0.98	0.98	0.00	0.00	0.41	0.00	1.00	1.00	0.00	0.00
-0.25	0.55	0.54	0.00	0.00	0.90	0.90	0.00	0.00	0.31	0.00	1.00	1.00	0.00	0.00
-0.20	0.41	0.40	0.00	0.00	0.76	0.76	0.00	0.00	0.24	0.00	0.98	0.98	0.00	0.00
-0.10	0.14	0.14	0.01	0.00	0.31	0.30	0.00	0.00	0.10	0.02	0.55	0.56	0.00	0.00
-0.00	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
0.10	0.01	0.01	0.22	0.22	0.00	0.00	0.34	0.34	0.02	0.15	0.00	0.00	0.62	0.62
0.20	0.00	0.00	0.49	0.50	0.00	0.00	0.78	0.78	0.01	0.33	0.00	0.00	0.97	0.97
0.25	0.00	0.00	0.67	0.67	0.00	0.00	0.91	0.91	0.00	0.43	0.00	0.00	1.00	1.00
0.30	0.00	0.00	0.80	0.81	0.00	0.00	0.98	0.98	0.00	0.54	0.00	0.00	1.00	1.00
0.40	0.00	0.00	0.94	0.95	0.00	0.00	1.00	1.00	0.00	0.78	0.00	0.00	1.00	1.00
0.45	0.00	0.00	0.97	0.98	0.00	0.00	1.00	1.00	0.00	0.84	0.00	0.00	1.00	1.00
0.49	0.00	0.00	0.99	0.99	0.00	0.00	1.00	1.00	0.00	0.90	0.00	0.00	1.00	1.00

NOTE: see Table 1. Critical values from Table 2. $>$: test against $d > 0$, $<$: test against $d < 0$.

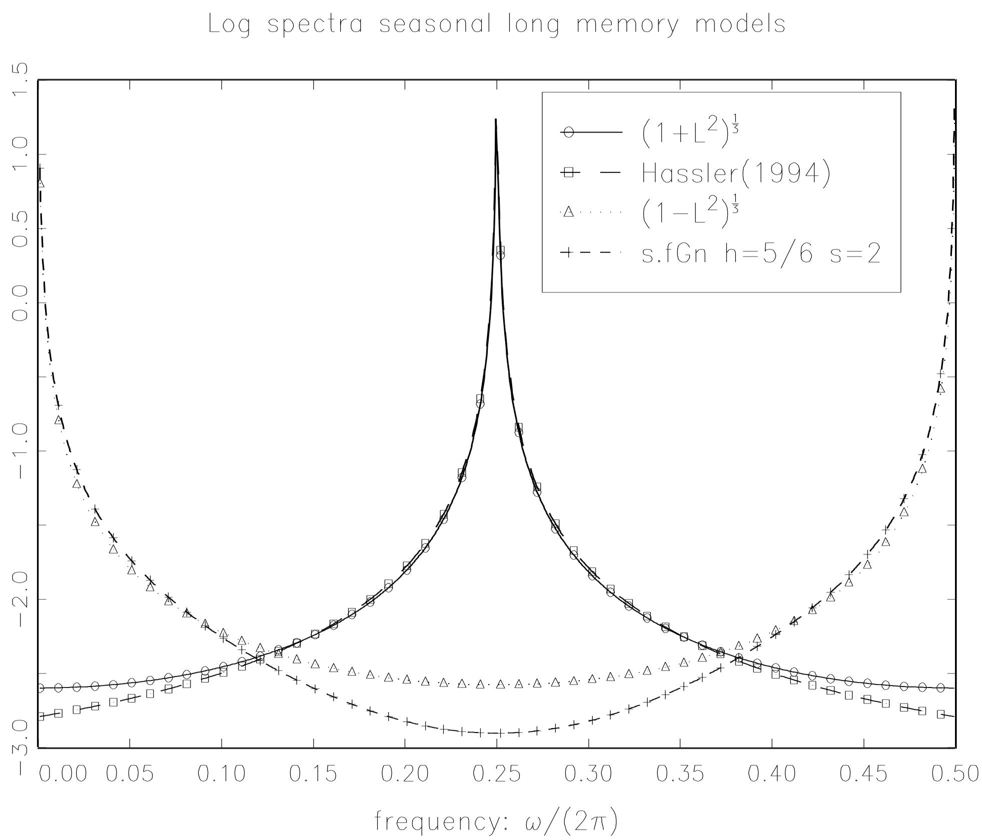

Figure 1: Log spectra from (5) for model with $D(z) = (1 + z^2)^{1/3}$, and its approximation by Hassler(1994), i.e. $X_i/3$ and $R_i/3$ from (19) and (20), the log spectra for the seasonally integrated model with $D(z) = (1 \quad x^2)^{1/3}$ and its corresponding seasonal fractional Gaussian noise process, see (7) with $h = d + 1/2 = 5/6$

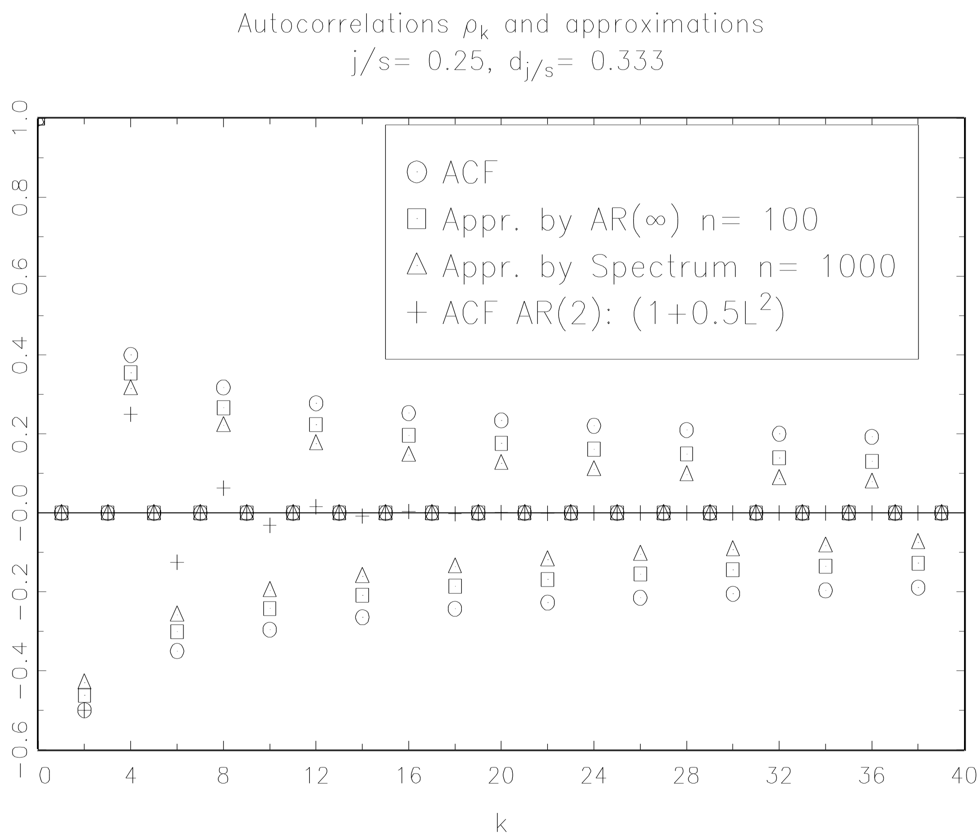

Figure 2: ACF of Seasonal long memory process $x_t' = (1 + L^2) \;\; ^{1/3} u_t$, with $\rho_2 = \;\; 0.5$, ACF of AR(∞) expansion of x_t' truncated at n=100, Approximation of the ACF from spectrum of x_t' using piecewise constant function at n=1000 points, and ACF of the short memory AR process $x_t'' = (1 + 0.5L^2) \;\; ^{1} u_t$.

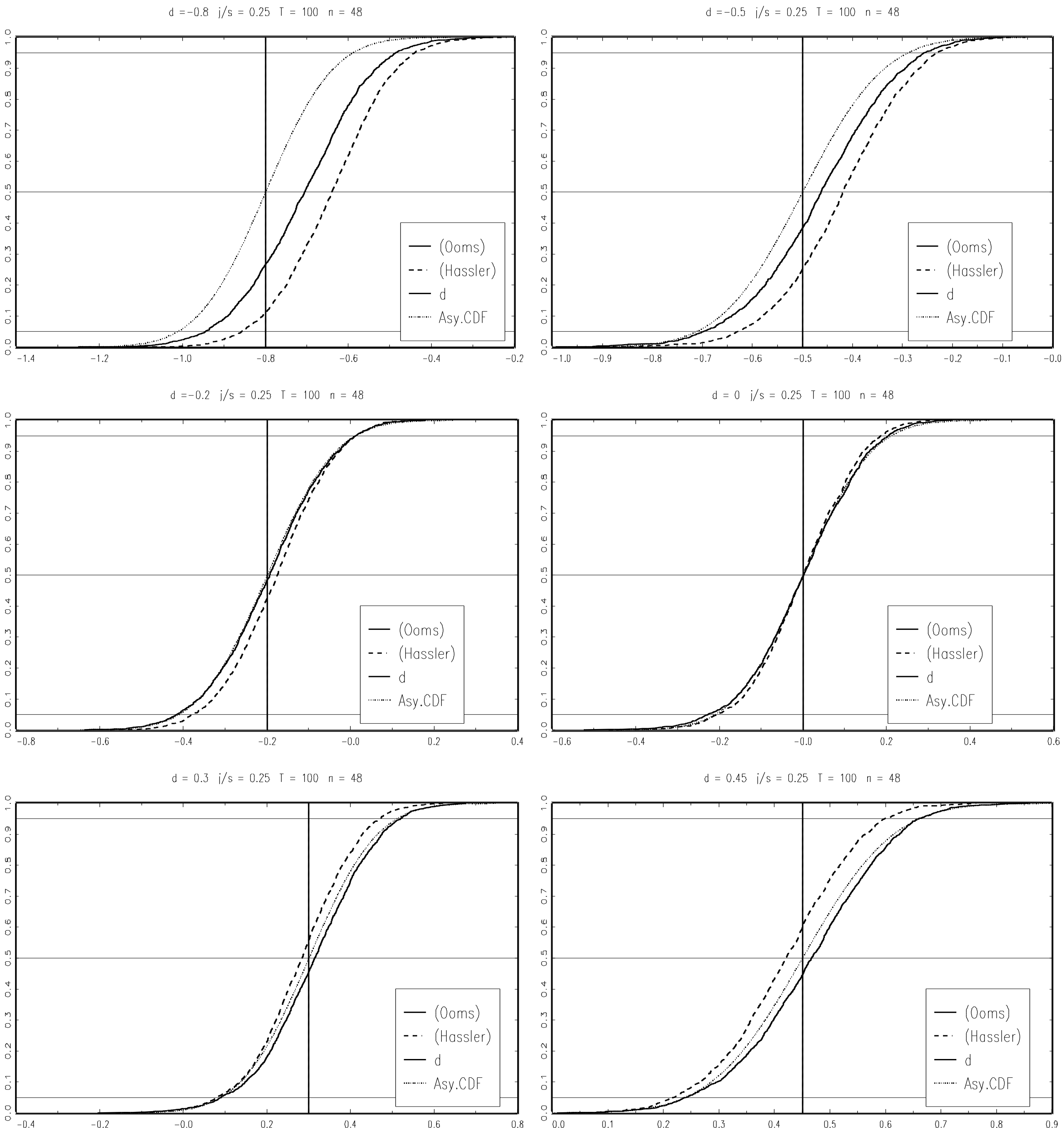

Figure 3: Empirical CDF log periodogram regression estimators of $d_{1/4}$ in $(1+L^2)^{d_{1/4}}x_t = u_t$. 100 observations. 48 periodogram points. Regressors used by Hassler (1994) and Ooms (1994) compared with CDF from asymptotic theory, $T = 100$, $d_{1/4} = 0.8, \; 0.5, \; 0.2, 0, 0.3, 0.45$. 2000 replications.

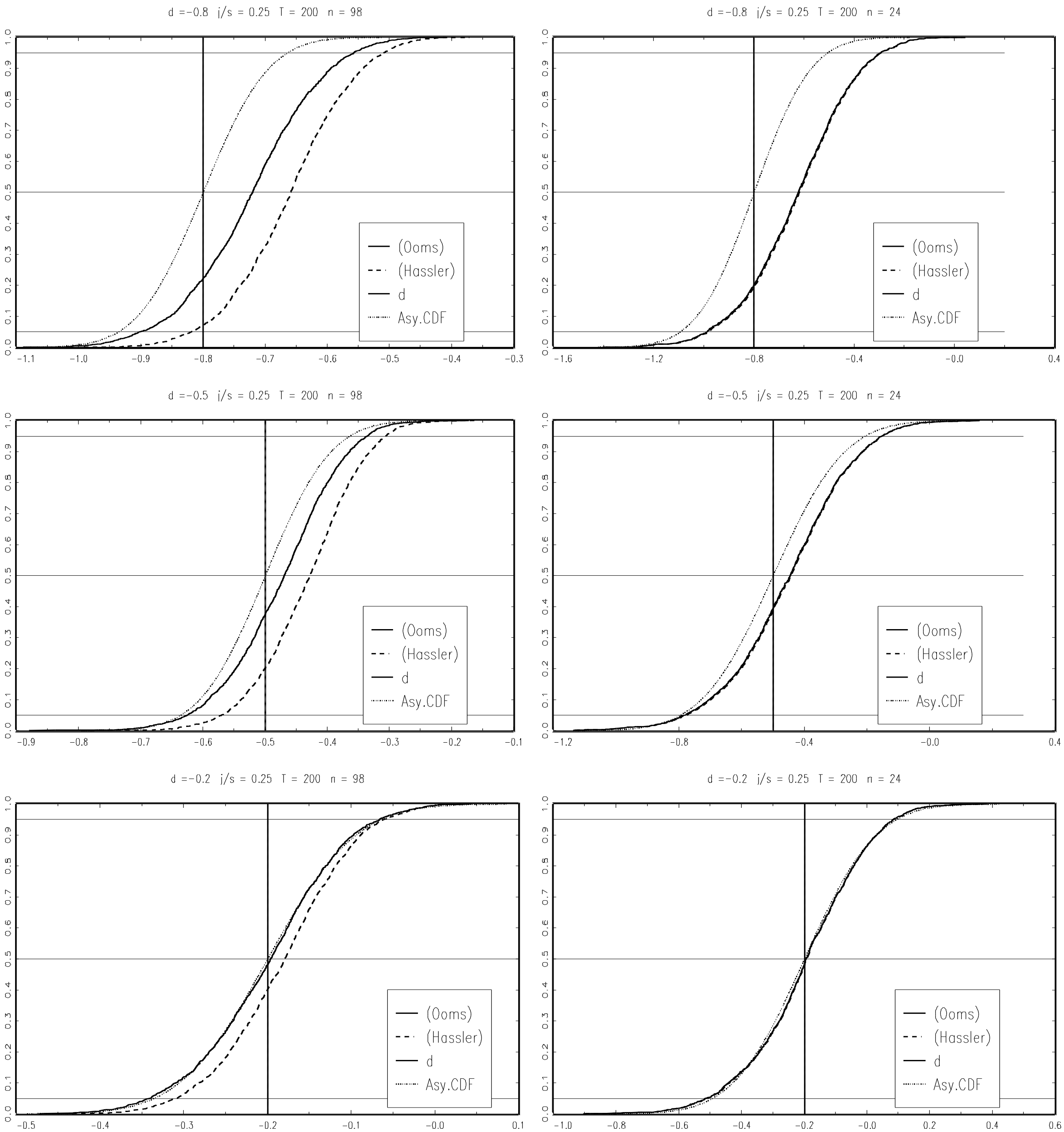

Figure 4: Empirical CDF log periodogram regression estimators of $d_{1/4}$ in $(1+L^2)^{d_{1/4}}x_t = u_t$. 200 observations, 98 (column left) and 24 periodogram points (column right). Negative orders of integration. Regressors used by Hassler (1994) and Ooms (1994) compared with CDF from asymptotic theory, $d_{1/4} =$ 0.8, 0.5, 0.2. 2000 replications.

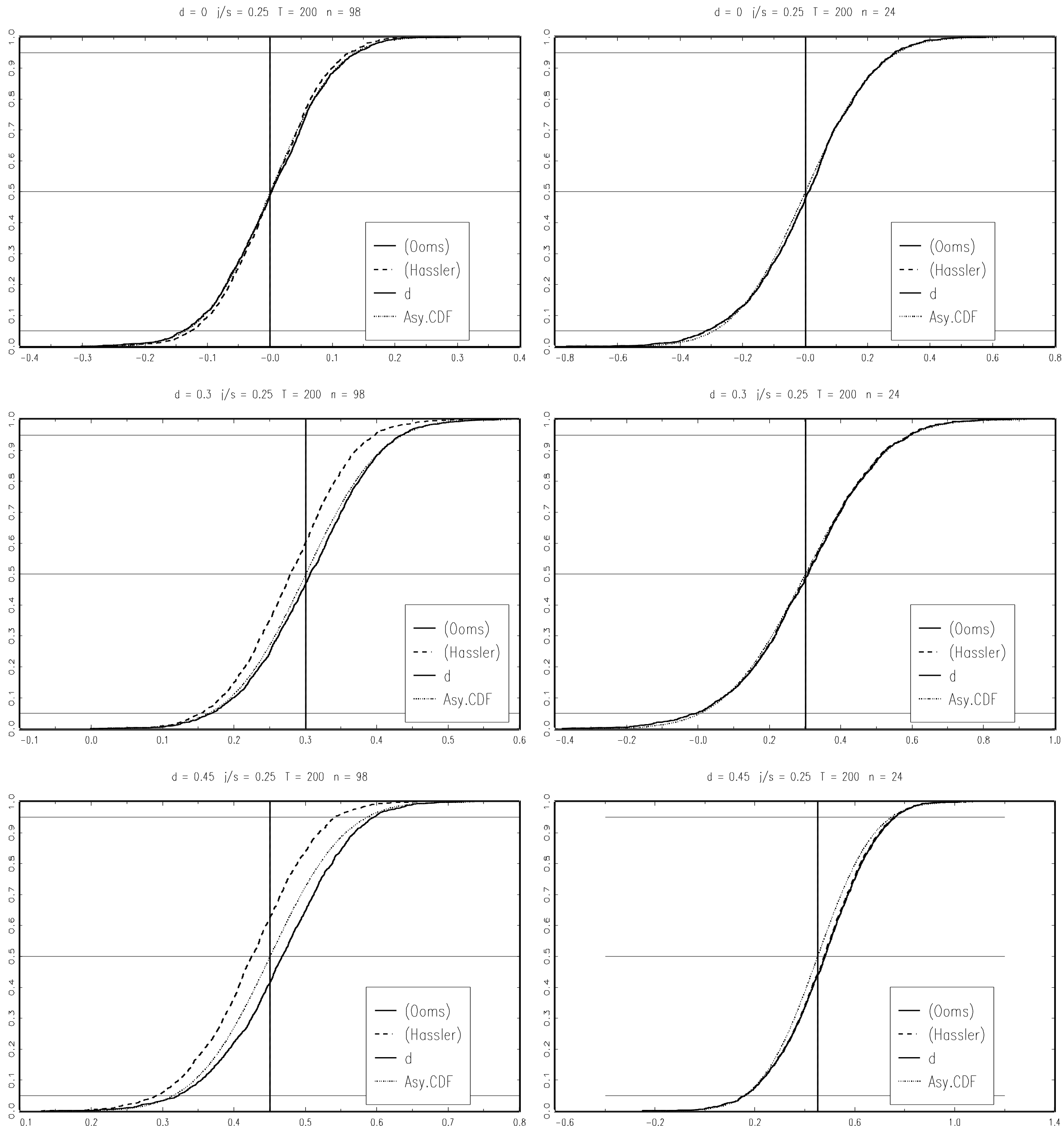

Figure 5: Empirical CDF log periodogram regression estimators of $d_{1/4}$ in $(1+L^2)^{d_{1/4}}x_t = u_t$. 200 observations. 98 (column left) and 24 periodogram points (column right). Positive orders of integration. Regressors used by Hassler (1994) and Ooms (1994) compared with CDF from asymptotic theory, $d_{1/4} = 0, 0.3, 0.45$. 2000 replications.

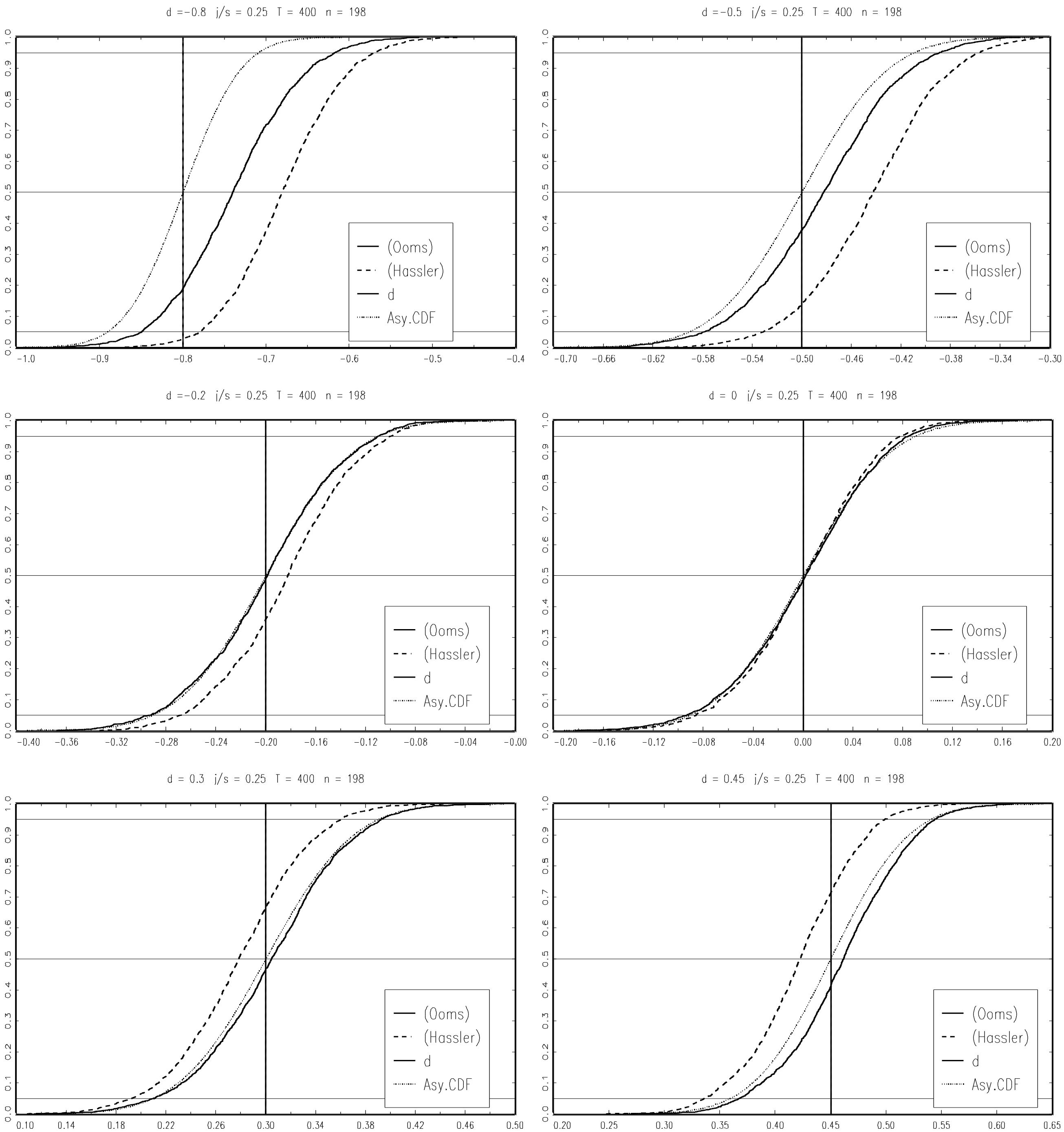

Figure 6: Empirical CDF log periodogram regression estimators of $d_{1/4}$ in $(1+L^2)^{d_{1/4}}x_t = u_t$. 400 observations. using 198 periodogram points. Regressors used by Hassler (1994) and Ooms (1994) compared with CDF from asymptotic theory, $T = 100$, $d_{1/4} =$ 0.8, 0.5, 0.2, 0, 0.3, 0.45. 2000 replications.

Table 4: *Estimation results approximate Frequency Domain ML and exact ML on quarterly change in log US GNP 1947.2-1989.4 and 1948.1-1989.4*

No.	d_0	$d_{1/4}$	$d_{1/2}$	ϕ_1	ϕ_2	ϕ_3	θ_1	θ_2	$\widehat{\sigma}_\epsilon$	$\lvert\lambda_p\rvert$	T
Ia	-0.70			-1.29	1.00	-0.54	-0.31	0.81	8.79	0.86	171
Ib	(0.24)			(0.26)	(0.24)	(0.14)	(0.11)	(0.12)			
IIa	-0.59			-1.18	0.93	-0.51	-0.29	0.81	9.50	0.81	171
IIb	(0.35)			(0.40)	(0.32)	(0.20)	(0.13)	(0.11)			
IIc	[0.25]			[0.32]	[0.32]	[0.20]	[0.21]	[0.17]			
IIIa	-0.45			-0.77					9.82	0.77	171
IIIb	(0.16)			(0.12)							
IVa	-0.69			-1.29	0.97	-0.51	-0.28	0.77	9.37	0.86	168
IVb	(0.26)			(0.29)	(0.28)	(0.15)	(0.13)	(0.13)			
Va	-0.85	-0.39	-0.28	-0.89					9.24	0.89	168
Vb	(0.19)	(0.09)	(0.11)	(0.08)							

NOTES: $\widehat{\sigma}_\epsilon$: asymptotic standard error residuals (in 1/1000) $|\lambda_p|$: absolute value of inverse of root of $\phi(z) = 0$ closest to 1. T: no. of observations Ia: Approximate Frequency Domain ML 47.2-89.4 using (14), Ib: asymptotic standard errors based on Hessian log-likelihood, IIa: Exact ML 47.2-89.4, Sowell(1992a), AIC optimal model IIb: asymptotic standard error, IIc: bootstrap standard deviations, IIIa and IIIb, see IIa and IIb (SIC optimal model) IVa: Approximate FDML 48.1-89.4, IVb: Asymptotic standard errors, Va: Approximate FDML 48.1-89.4. Flexible Seasonal ARFIMA model Vb: Asymptotic standard errors.

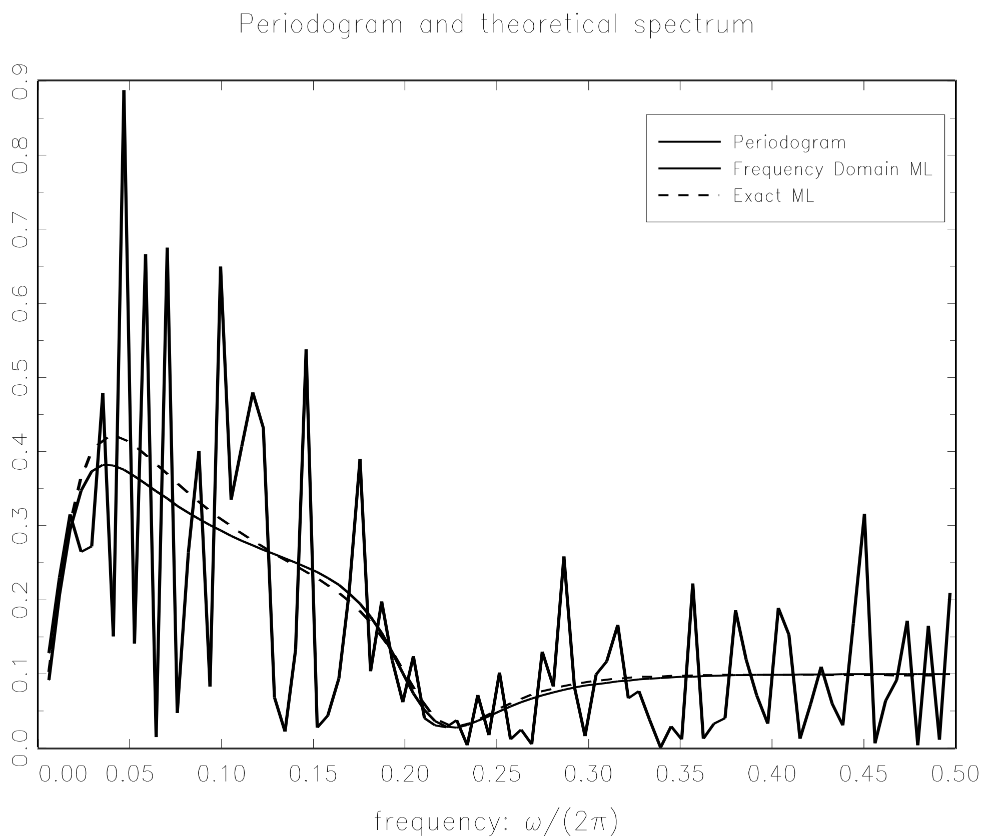

Figure 7: Periodogram change log US GNP 47.II-89.IV and spectra ARFIMA(3,d,2) models fit by approximate frequency domain ML and exact ML.

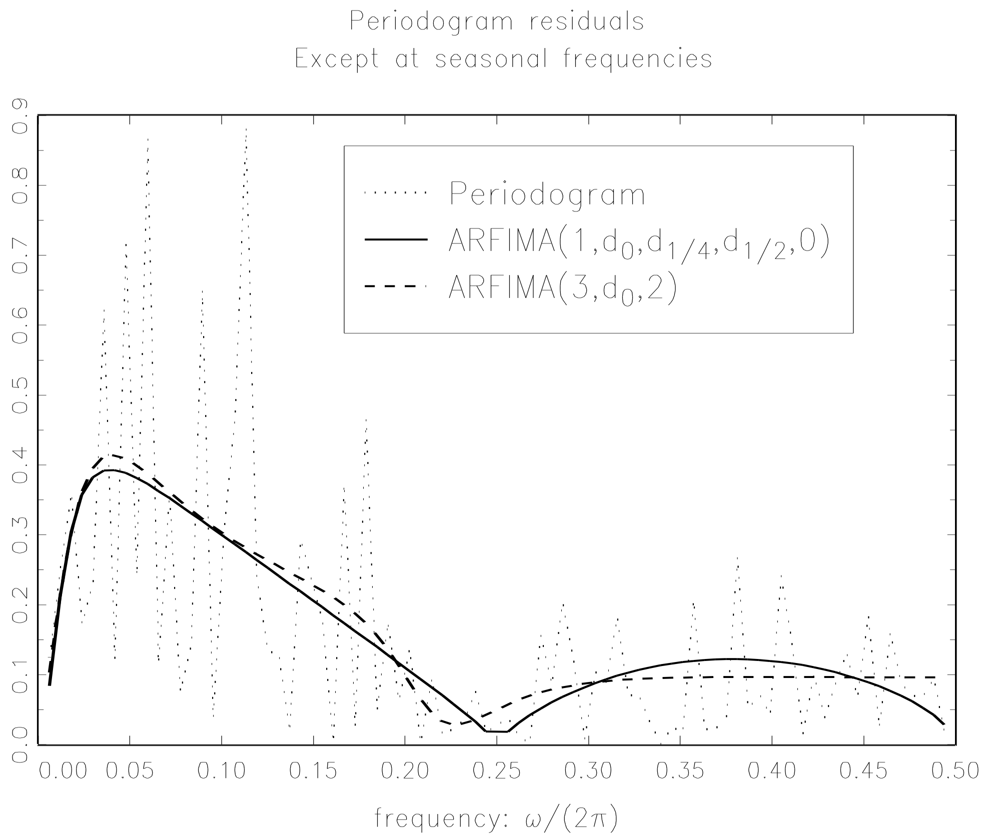

Figure 8: Periodogram change log US GNP 48.I-89.IV, spectrum flexible ARFIMA$(1, d, 0)_s$ model fit by approximate frequency domain ML and spectrum ARFIMA(3,d,2) model fit by exact ML for period 47.II-89.IV.

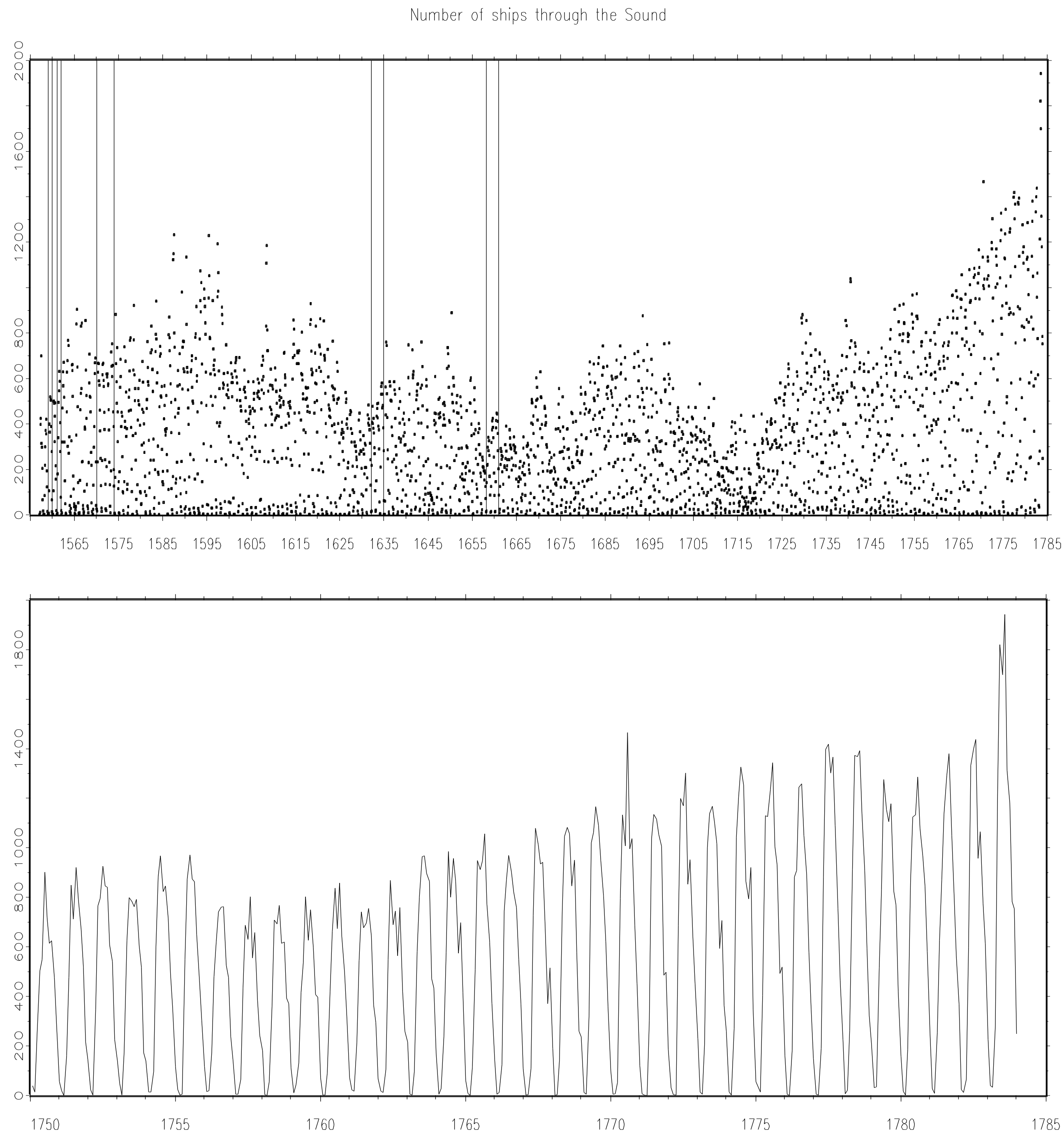

Figure 9: Monthly number of ships registered passing Sound 1557-1783, Vertical lines indicate beginning and end of period without observations. Values for these periods estimated using backcasting and forecasting from periodic AR model. Lower panel time series plot indicates relatively smooth intra-year seasonal pattern, and persistence of low values in winter months.

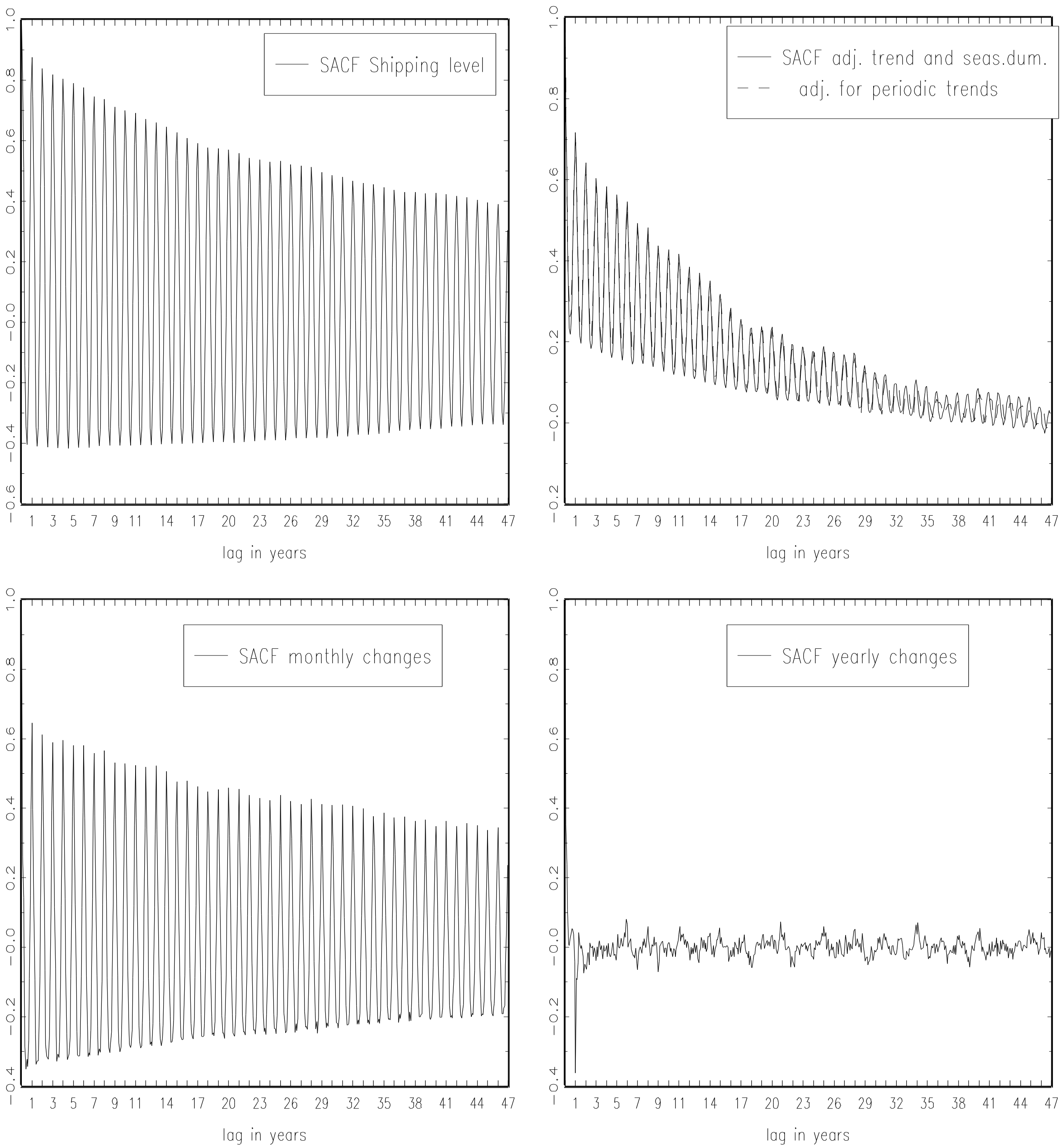

Figure 10: Sample autocorrelation functions of number of ships. 1. Levels. 2. Levels adjusted for linear deterministic trend and seasonal dummies together with levels corrected for seasonal deterministic trends, (difference hard to notice). 3. First differences. 4. Yearly differences. Plots 1. 2. and 3. exhibit clear long memory. Plot 4. shows signs of "overdifferencing" or anti-persistence.

Table 5: *Estimation results approximate Frequency Domain ML and Periodogram Regressions on month-to-month change in number of ships through Sound 1558.1-1783.12*

	FDML			Periodogram regression			
				$n = 1356$		$n = 57$	
Parameter	Estim.	St.error	t-value	Estim.	St.error	Estim.	St.error
d_0	-0.40	0.063	6.43*	-0.55	0.037	-0.60	0.109
$d_{1/2}$	0.04	0.106	0.41	0.24	0.037	0.10	0.109
$d_{1/12}$	0.49	0.041	12.10*	0.36	0.031	0.62	0.125
$d_{1/6}$	0.35	0.038	9.10*	0.20	0.031	0.47	0.125
$d_{1/4}$	0.26	0.039	6.74*	0.13	0.031	0.13	0.125
$d_{1/3}$	0.17	0.046	3.77*	0.23	0.031	0.26	0.125
$d_{5/12}$	0.10	0.060	1.64	0.19	0.031	-0.04	0.125
ϕ_1	0.96	0.218	4.40*	-	-	-	-
ϕ_2	0.29	0.192	1.51	-	-	-	-
ϕ_3	-0.04	0.098	-0.38	-	-	-	-
ϕ_4	-0.02	0.068	-0.27	-	-	-	-
ϕ_5	0.04	0.055	0.79	-	-	-	-
ϕ_6	-0.00	0.054	-0.07	-	-	-	-
ϕ_7	-0.02	0.053	-0.44	-	-	-	-
ϕ_8	-0.00	0.039	-0.07	-	-	-	-
ϕ_9	-0.01	0.037	-0.24	-	-	-	-
ϕ_{10}	-0.10	0.033	2.94*	-	-	-	-
ϕ_{11}	-0.09	0.032	2.77*	-	-	-	-
ϕ_{12}	-0.02	0.031	-0.69	-	-	-	-

Seasonal means monthly changes (OLS estimates):

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
-76	-1	147	313	119	18	-11	-58	-76	-91	-142	-141

Roots AR-part:

no. k	Inverse root: z_k^{1}	$\left\lvert z_k^{1} \right\rvert$	frequency	"UR-test"
1	-0.43	0.43	0.50	-1538
2	-0.58	0.58	0.50	-1142
3	-0.73	0.73	0.50	-734
4	-0.60-0.48i	0.77	0.39	-1087
5	-0.60+0.48i	0.77	0.39	-1087
6	0.24-0.74i	0.78	0.20	-1067
7	0.24+0.74i	0.78	0.20	-1067
8	0.63-0.47i	0.79	0.10	-1024
9	0.63+0.47i	0.79	0.10	-1024
10	0.79	0.79	0.00	-563
11	-0.28-0.75i	0.80	0.31	-958
12	-0.28+0.75i	0.80	0.31	-958

NOTES: Total no. of parameters: 31, 12 seasonal means, 7 long memory parameters $d_{j/s}$ and 12 AR parameters ϕ_k. Periodogram regression $n = 1356$: Use all Fourier frequencies ($\omega_i = 2\pi i/T,\ i = 1, \ldots, T/2$), except $\omega_i = 2\pi j/(12T),\ j = 0, 1, \ldots, 6$ and immediately neigbouring frequencies. $n = 57$: use 57 frequencies in neigborhood frequency $2\pi j/12$, except the frequency itself and the closest neigboring frequency/frequencies. Exclude one extra periodogram ordinate for d_0 and $d_{1/2}$, exclude two extra ordinates for the other integration parameters. "UR-test" statistic for $|z_k| = 1$: $T(1\ \ \big|z|^{l}\big|)$, with $l = 1$ for real z and $l = 2$ for complex z, approximate 5% critical values -14.1 and -19.0, given $d_{j/s} = 0,\ j = 0, 1, \ldots, 6$.

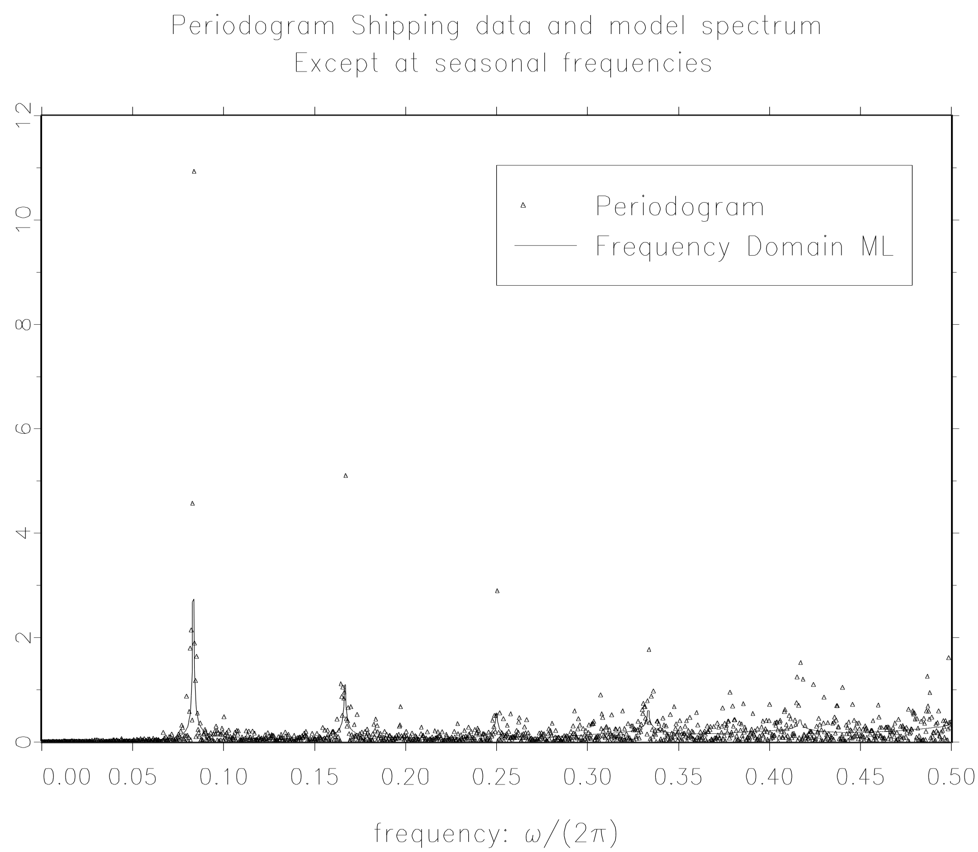

Figure 11: Periodogram month-to-month changes in shipping 1558.1-1783.12 and spectrum ARFIMA$(12, d_0, \ldots, d_{1/2}, 0)_{12}$ model estimated by approximate frequency domain ML.

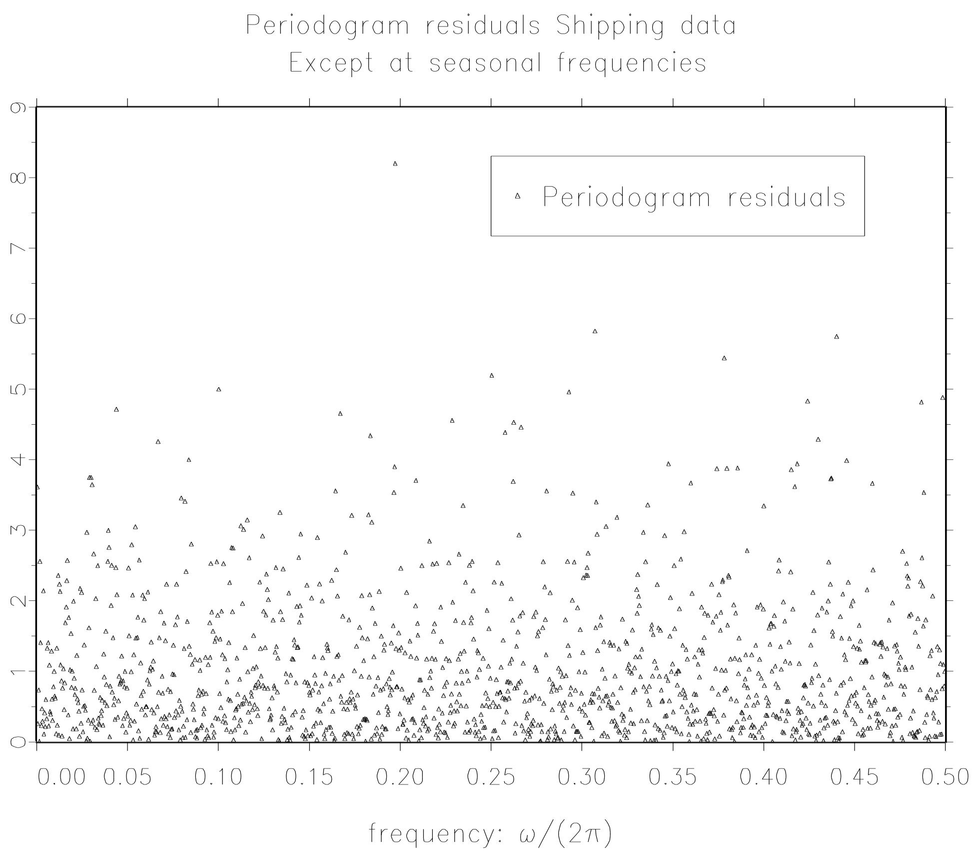

Figure 12: Periodogram Residuals month-to-month changes in shipping 1558.1-1783.12 of ARFIMA$(12, d_0, \ldots, d_{1/2}, 0)$ model estimated by approximate frequency domain ML.

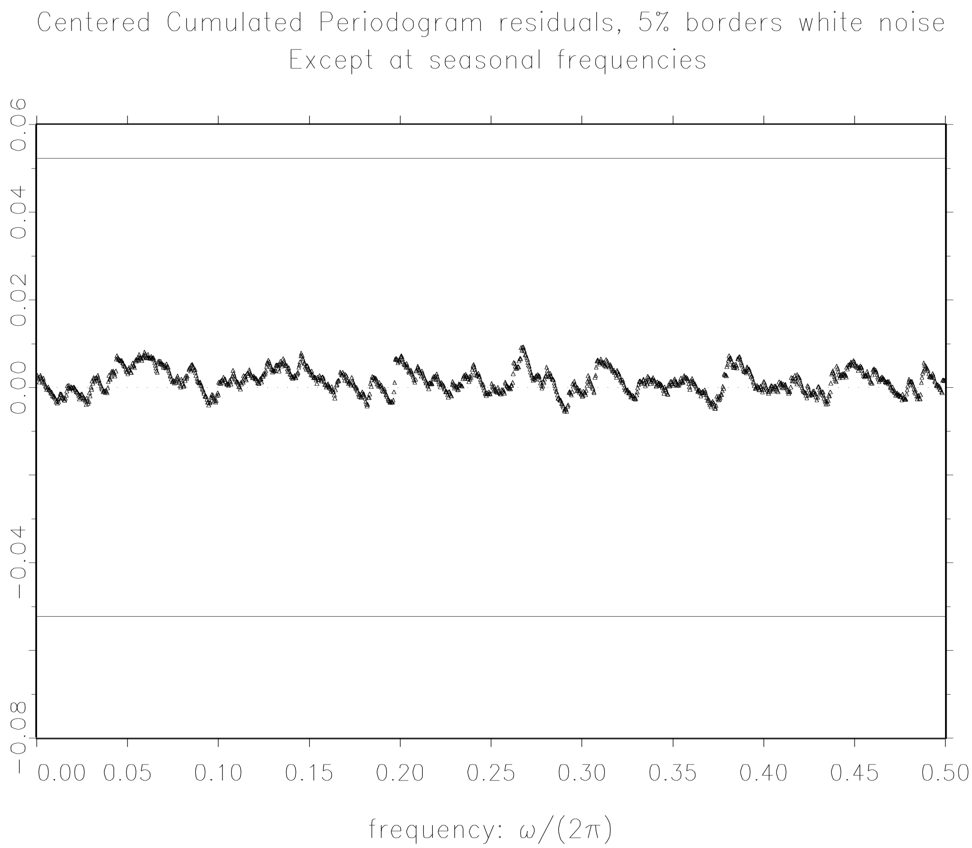

Figure 13: Centered Cumulative Periodogram Residuals month-to-month in shipping 1558.1-1783.12: D_l defined in equation (23).

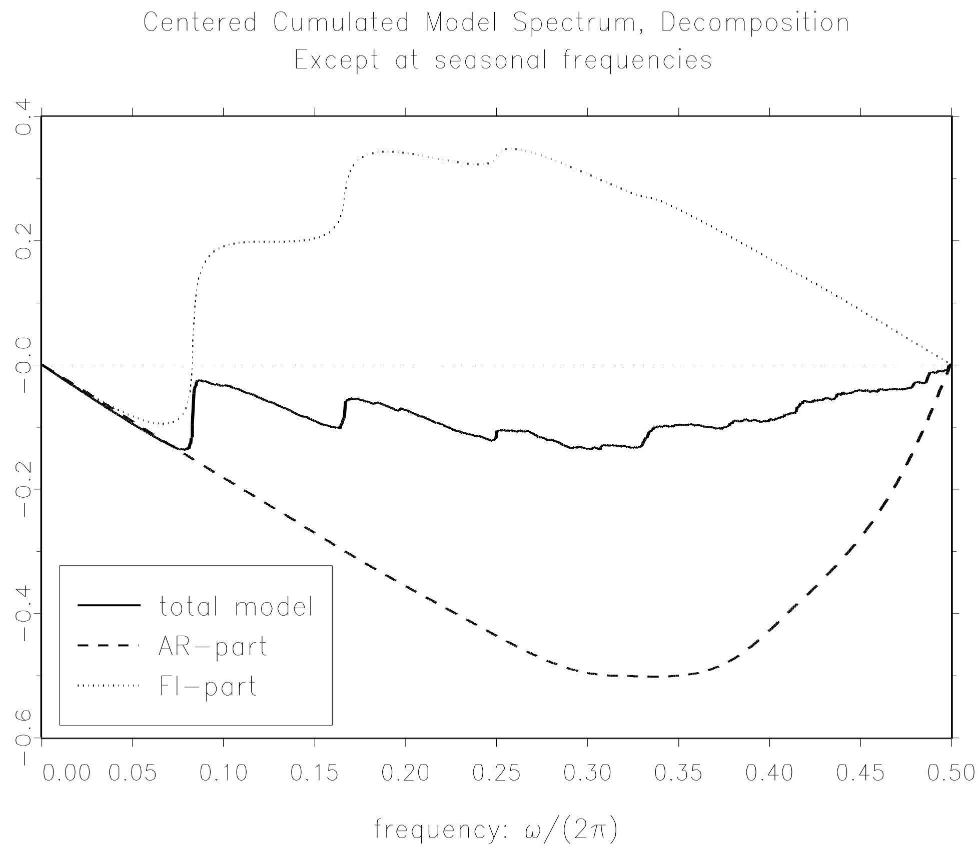

Figure 14: Centered Cumulative Model Spectrum for total model and its components.

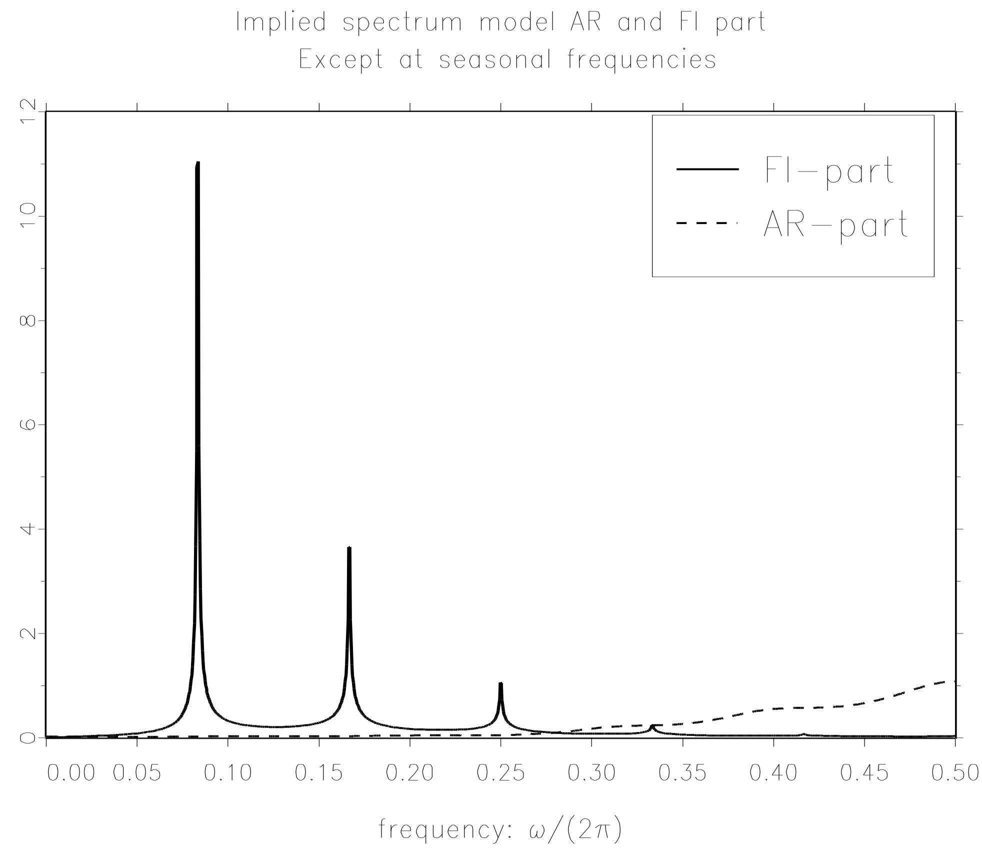

Figure 15: Shipping data: Model spectrum decomposition, Long memory FI-part $|D(e^{i\omega_j})|^2$ and Weakly dependent AR-part $|\phi(e^{i\omega_j})|^2$.

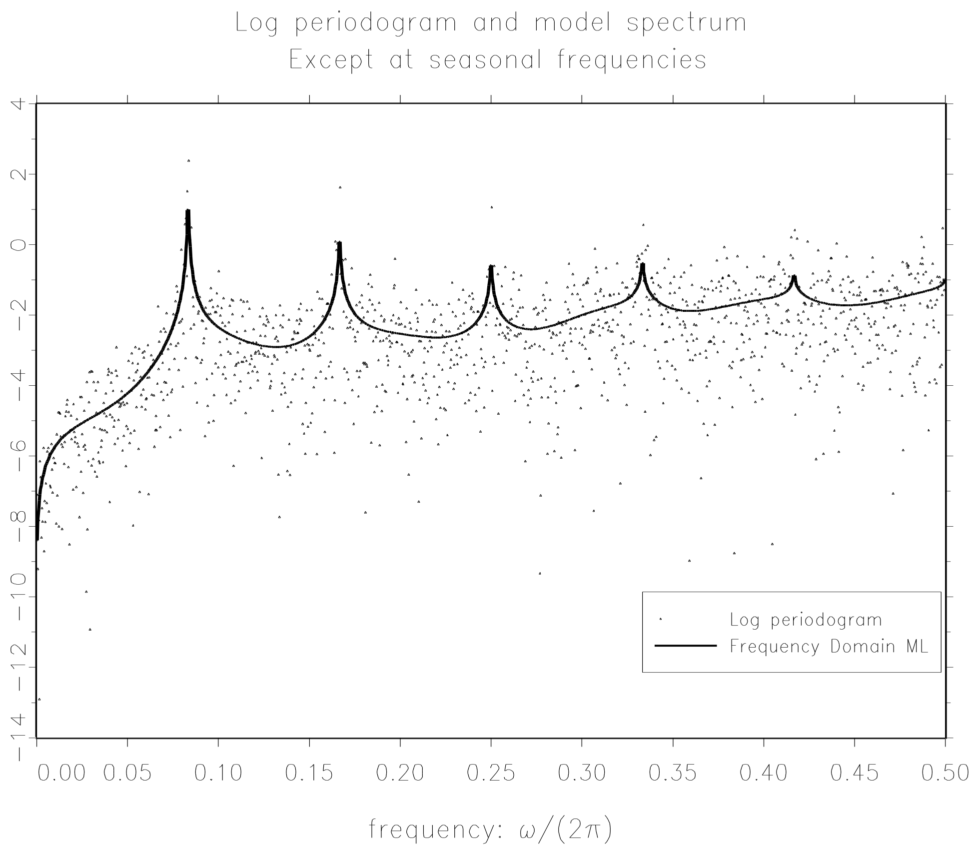

Figure 16: Shipping data: Log periodogram and log model spectrum.